PROBLEMS TO BE SOLVED IN CLASSROOM

The asterisk gives idea about the difficulty level: * Some difficulty: ** Advanced level

Unit 0. Prerrequisites

0.1. Obtain a unit vector perpendicular to vectors $2\vec{i}+3\vec{j}-6\vec{k}$ and $\vec{i}+\vec{j}-\vec{k}$

*Solution:*

$$(2\vec{i}+3\vec{j}-6\vec{k})\times(\vec{i}+\vec{j}-\vec{k})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & -6\\ 1 & 1 & -1\end{vmatrix}=3\vec{i}-4\vec{j}-\vec{k}$$

$$\left|3\vec{i}-4\vec{j}-\vec{k}\right|=\sqrt{3^2+4^2+1^2}=\sqrt{26} \qquad \vec{u}=\pm\frac{3\vec{i}-4\vec{j}-\vec{k}}{\sqrt{26}}$$

0.2 a) Find the integral of vector $\vec{v}=2xy\vec{i}+3\vec{j}-2z^2\vec{k}$ along the straight line parallel to Y axis from point A(1,1,1) to point B(1,3,1) (circulation of a vector along a line).

b) Repeat the above exercise with vector $\vec{v}=2xy\vec{i}+3x\vec{j}-2z^2\vec{k}$ .

c) If possible, repeat the exercise b) but along the straight line going from A to point C (3,3,1).

*Solution:*

a) Line going from A to B is a straight line parallel to Y axis. Then $d\vec{r}=dy\vec{j}$

$$\int_A^B \vec{v}d\vec{r}=\int_A^B(2xy\vec{i}+3\vec{j}-2z^2\vec{k})dy\vec{j}=\int_1^3 3dy=3(3-1)=6$$

b) $\int_A^B \vec{v}d\vec{r}=\int_A^B(2xy\vec{i}+3x\vec{j}-2z^2\vec{k})dy\vec{j}=\int_1^3 3xdy$

As x is constant (x=1) along the line AB, then $\int_1^3 3xdy=\int_1^3 3dy=3(3-1)=6$

c) Now, the line going from A to C is not parallel to Y axis, and then $d\vec{r}=dx\vec{i}+dy\vec{j}$ . Its equation is x=y. Therefore

$$\int_A^C \vec{v}d\vec{r}=\int_A^C(2xy\vec{i}+3x\vec{j}-2z^2\vec{k})(dx\vec{i}+dy\vec{j})=\int_A^C(2xydx+3xdy)=\int_1^3 2x^2dx+\int_1^3 3ydy=$$

$$=2\frac{x^3}{3}\bigg|_1^3+3\frac{y^2}{2}\bigg|_1^3=\frac{2}{3}26+\frac{3}{2}8=\frac{104+72}{6}=\frac{176}{6}=\frac{88}{3}$$

0.3 Find the integral of vector $\vec{v}=2xy\vec{i}+3\vec{j}-2z^2\vec{k}$ through a square with side *h* parallel to the plane XY and placed on plane *z=1* (integral of a vector through a surface).

*Solution:*

The vector surface of square will be $d\vec{S}=dS\vec{k}$ (it could be also taken $d\vec{S}=-dS\vec{k}$ )

$$\int_{square}\vec{v}d\vec{S}=\int_{square}(2xy\vec{i}+3\vec{j}-2z^2\vec{k})dS\vec{k}=-\int_{square}2z^2dS$$

As the square lays on plane z=1 $-\int_{square} 2z^2 dS = -\int_{square} 2dS = -2\int_{square} dS = -2S_{square} = -2h^2$

If $d\vec{S} = -dS\vec{k}$ had been taken, then $\int_{square} \vec{v}d\vec{S} = 2h^2$ Both results are valid

0.4 Calculate:

a) The circulation of vector $\vec{v} = r\vec{u}_r$ along any circumference having radius *2* and centred at the origin of coordinates.

b) The integral of vector $\vec{v} = r\vec{u}_r$ through a sphere having radius *2* and centred at the origin of coordinates.

c) The circulation of vector $\vec{v} = r\vec{u}_r$ along any radius of before sphere between points with *r=0* and *r=2*.

*Solution:*

a) Vector $\vec{v}$ goes in the direction of radius of circumference and then it is perpendicular to the tangent of circumference at any point. So

$$\int_{circumference} \vec{v}d\vec{r} = 0$$

b) Now, vector $\vec{v}$ is always parallel to surface vector of sphere ($d\vec{S} = dS\vec{u}_r$). Moreover r=2 and so

$$\int_{sphere} \vec{v}d\vec{S} = \int_{sphere} r\vec{u}_r dS\vec{u}_r = \int_{sphere} rdS = 2\int_{sphere} dS = 2 \cdot 4\pi 2^2 = 32\pi$$

c) $$\int_{radius} \vec{v}d\vec{r} = \int_{radius} r\vec{u}_r dr\vec{u}_r = \int_0^2 rdr = \left.\frac{r^2}{2}\right|_0^2 = 2$$

Unit 1: Electrostatics

1.1 a) Calculate the electric field produced at point (4,0) m by two point charges: 2 µC at (0,0) m and -2 µC at (0,3) m. Calculate the force acting over -5 µC placed at point (4,0) m.

b) Repeat the before exercise but at point (4,4) m instead point (4,0) m.

*Solution:*

a) $\vec{E}_2 = k\frac{q}{r^2}\vec{u}_r = 9\cdot 10^9\frac{2\cdot 10^{-6}}{4^2}\vec{i} = \frac{9}{8}10^3\vec{i}\ N/C$

$$\vec{E}_{-2} = k\frac{q}{r^2}\vec{u}_r = 9\cdot 10^9\frac{2\cdot 10^{-6}}{5^2}\frac{-4\vec{i}+3\vec{j}}{5} = \frac{18}{125}10^3(-4\vec{i}+3\vec{j})\,N/C$$

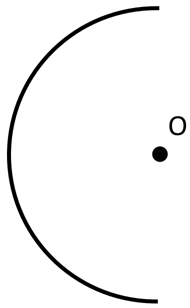
$$\vec{E}_{(4,0)} = \vec{E}_2+\vec{E}_{-2} = \frac{9}{8}10^3\vec{i} + \frac{18}{125}10^3(-4\vec{i}+3\vec{j}) = 549\vec{i}+432\vec{j}\ N/C$$

$$\vec{F}_{-5} = -5\cdot 10^{-6}\vec{E} = -5\cdot 10^{-6}(549\vec{i}+432\vec{j}) = -(2745\vec{i}+2160\vec{j})\cdot 10^{-6}\ N$$

b) $\vec{E}_{(4,4)} = k\frac{2\cdot 10^{-6}}{32}\frac{(\vec{i}+j)}{\sqrt{2}} + k\frac{2\cdot 10^{-6}}{17}\frac{(-4\vec{i}-j)}{\sqrt{17}} = -629\vec{i}+141\vec{j}\ N/C$

$$\vec{F}_{-5} = -5\cdot 10^{-6}\vec{E} = -5\cdot 10^{-6}(-629\vec{i}+141\vec{j}) = (3145\vec{i}-705\vec{j})\cdot 10^{-6}\ N$$

1.2 ** A uniform charged rod of length 14 cm is bent into the shape of a semicircle as shown on picture. The rod has a total charge of -7,5 µC, homogeneously distributed along the rod. Find the magnitude and the direction of the electric field at point O, the centre of the semicircle.

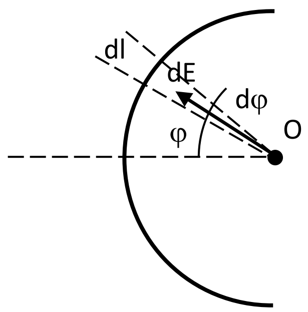



*Solution:*

On this case, the charge is uniformly distributed along the semicircle. Its linear distribution of charge is: $\lambda = \frac{Q}{\ell} = -\frac{7,5\cdot 10^{-6}}{14\cdot 10^{-2}} = -0,5357\cdot 10^{-4}C/m$

The radius of semicircle is:

$$R = \frac{\ell}{\pi} = \frac{14\cdot 10^{-2}}{\pi} = 4,46\cdot 10^{-2}m$$




We'll solve the problem with letters, and at the end we'll change the letters by numbers. If we consider a differential element of semicircle (length dl), the charge on this element is: $dq = \lambda d\ell$

The electric field produced by this infinitesimal element at point O is: $dE = k\frac{\lambda dl}{R^2}$

being its direction that joining the infinitesimal element and O.

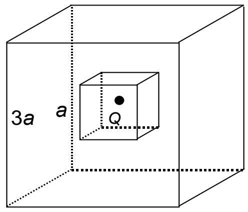
If we consider an infinitesimal element of semicircle symmetric to that on picture, the resulting field of both elements is an horizontal electric field pointing to left whose magnitude is:

$$dE_h = 2k\frac{\lambda dl}{R^2}cos\varphi$$

And the total electric field at point O is the integral of every infinitesimal elements of semicircle, by taking in account that $d\ell = Rd\varphi$, is:

$$E_h = \int_0^{\frac{\pi}{2}} 2k\frac{\lambda Rd\varphi}{R^2}cos\varphi = \frac{2k\lambda}{R}\int_0^{\frac{\pi}{2}} cos\varphi d\varphi = \frac{2k\lambda}{R} = 2{,}16\cdot 10^7\ N/C$$

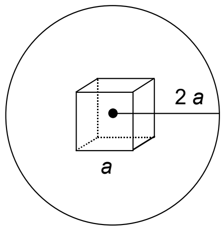
1.3 Let's take the point charge *Q* and the two cubic surfaces, parallel, centered in *Q*, and with sides *a* and 3*a*, shown in the figure. Compute the rate between fluxes of the electric field through both surfaces (Φ*a*/Φ3*a*). Justify the answer.




*Solution:*

$$\Phi_a = \frac{Q}{\varepsilon_0} \qquad \Phi_{3a} = \frac{Q}{\varepsilon_0} \qquad \frac{\Phi_a}{\Phi_{3a}} = 1$$

1.4 A cube of edge *a* and uniform volumetric density of charge, ρ, is placed on vacuum. It's surrounded by a spherical surface of radius 2*a*. Compute the flux of the electric field through the spherical surface.




*Solution:*

$$\Phi_{2a} = \frac{Q}{\varepsilon_0} = \frac{\rho a^3}{\varepsilon_0}$$

1.5 Calculate the electric field produced by:

a) A spherical surface having radius *R* charged with a homogeneous density of charge $\sigma$; calculate the electric field at a distance *r* from the centre of sphere for r<R and r>R.

b) An infinite plane charged with $\sigma$ homogeneous.

c) Two infinite and parallel planes charged with a homogeneous density of charge $\sigma$, inside the space between both planes and outside the space (applying superposition). Also consider the case when the charges on both planes have different sign.

d) The same exercise as a) but adding a new and negative point charge Q at the centre of sphere. ¿Could be the electric field zero at any point of the space? ¿In which circumstances?

*Solution:*

a) The charge is distributed over the surface, without any charge inside. Therefore, applying Gaus's law to a sphere with radius r:

r<R $\qquad \Phi = \int_{sphere} \vec{E}d\vec{S} = E4\pi r^2 = \frac{0}{\varepsilon_0} \Rightarrow E = 0$

r>R $\qquad \Phi = \int_{sphere} \vec{E}d\vec{S} = E4\pi r^2 = \frac{\sigma 4\pi R^2}{\varepsilon_0} \Rightarrow E = \frac{\sigma R^2}{\varepsilon_0 r^2}$

b) $E = \frac{\sigma}{2\varepsilon_0}$

c) If both charges have the same sign:

Inside the space between both planes: $E = 0$

Outside the space between both planes: $E = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$

If both charges have different sign:

Inside the space between both planes: $E=\frac{\sigma}{2\varepsilon_0}+\frac{\sigma}{2\varepsilon_0}=\frac{\sigma}{\varepsilon_0}$

Outside the space between both planes: $E=0$

d) r<R $$\Phi=\int_{sphere}\vec{E}d\vec{S}=E4\pi r^2=\frac{Q}{\varepsilon_0}\Rightarrow E=\frac{Q}{4\pi\varepsilon_0 r^2}$$

r>R $$\Phi=\int_{sphere}\vec{E}d\vec{S}=E4\pi r^2=\frac{\sigma 4\pi R^2+Q}{\varepsilon_0}\Rightarrow E=\frac{\sigma R^2}{\varepsilon_0 r^2}+\frac{Q}{4\pi\varepsilon_0 r^2}$$

Electric field can only be zero if $Q=-4\pi\sigma R^2$. In this case, electric field is zero at any point outside the sphere.

1.6 Calculate:

a) The electric potential at points *A (4,0)* and *B (4,3)* produced by the charges of exercise 1.1: $2\ \mu C$ at *(0,0)* and *-2 μC* at *(0,3)*.

b) The work needed to carry a charge of *-3 μC* from *A* to *B*.

*Solution:*

a) $$V_A=9\cdot 10^9\frac{2\cdot 10^{-6}}{4}-9\cdot 10^9\frac{2\cdot 10^{-6}}{5}=18\cdot 10^3(\frac{1}{4}-\frac{1}{5})=900\ V$$

$$V_B=9\cdot 10^9\frac{2\cdot 10^{-6}}{5}-9\cdot 10^9\frac{2\cdot 10^{-6}}{4}=18\cdot 10^3(\frac{1}{5}-\frac{1}{4})=-900\ V$$

b) $$W=q(V_A-V_B)=-3\cdot 10^{-6}(900+900)=-0{,}0054\ J$$

1.7 Calculate the d.d.p. (difference of potential) between two points A and B (being d the distance between A and B perpendicullary measured to the plane) of the electric field created by an infinite plane charged with surface density of charge σ.

*Solution:*

$$V_A-V_B=\int_A^B\vec{E}d\vec{r}=Ed=\frac{\sigma d}{2\varepsilon_0}$$

1.8 A spherical surface (radius *R*) is charged with a homogeneous surface density of charge $\sigma$ (exercise 1.5). Calculate:

a) Difference of potential between A(2R) and B(3R).
b) Electric potential on point B.
c) Difference of potential between C(R/3) and D(R/2).
d) Electric potential on points C and D.
e) Work done by the electric field to carry a charge *q* from B to A.
f) Work done by the electric field to carry a charge *q* from A to E(2R) (being E a point different than A).
g) If σ is positive and we add a negative point charge *Q* at the center of sphere, ¿is possible to find a point where the electric field was zero?

*Solution:*

a) In order to apply Gaus's law, we'll consider spherical surfaces for different r:

r<R $\qquad E\cdot 4\pi r^2 = \dfrac{0}{\varepsilon_0} \Rightarrow E = 0$

r>R $\qquad E\cdot 4\pi r^2 = \dfrac{\sigma 4\pi R^2}{\varepsilon_0} \Rightarrow E = \dfrac{\sigma R^2}{\varepsilon_0 r^2}$

b) $V_A - V_B = \int_{2R}^{3R} \vec{E}d\vec{r} = \int_{2R}^{3R} \dfrac{\sigma R^2}{\varepsilon_0 r^2}dr = \dfrac{\sigma R^2}{\varepsilon_0}\dfrac{1}{6R} = \dfrac{\sigma R}{6\varepsilon_0}$

c) $V_B = \int_{3R}^{\infty} \vec{E}d\vec{r} = \int_{3R}^{\infty} \dfrac{\sigma R^2}{\varepsilon_0 r^2}dr = \dfrac{\sigma R^2}{\varepsilon_0}\dfrac{1}{3R} = \dfrac{\sigma R}{3\varepsilon_0}$

d) Points C and D are inside the spherical surface, where the electric field is null and the electric potential constant. Then $V_C - V_D = \int_{R/3}^{R/2} \vec{E}d\vec{r} = \int_{R/3}^{R/2} 0d\vec{r} = 0$

e) As electric potential inside the spherical surface is constant and the electric potential is a continuous function, the electric potential at any point inside the spherical surface will be equal to the potential at any point over the surface, where r=R:
$V_C = V_D = \int_{R}^{\infty} \vec{E}d\vec{r} = \int_{R}^{\infty} \dfrac{\sigma R^2}{\varepsilon_0 r^2}dr = \dfrac{\sigma R^2}{\varepsilon_0}\dfrac{1}{R} = \dfrac{\sigma R}{\varepsilon_0}$

f) $W_{BA} = q(V_B - V_A) = -\dfrac{q\sigma R}{6\varepsilon_0}$

g) As A and E are points with equal radii, they lie over an equipotential surface, and then the work needed to carry any charge from A to E is null: $W_{AE} = q(V_A - V_E) = 0$

h) If we add a charge Q at the centre of spherical surface, then the electric field at any point outside of spherical surface will be (applying Gaus's law):

$$E\cdot 4\pi r^2 = \frac{\sigma 4\pi R^2 + Q}{\varepsilon_0} \Rightarrow E = \frac{\sigma 4\pi R^2 + Q}{4\pi\varepsilon_0 r^2}$$

The electric field is not null in general. The only possibility to get electric field zero is that $\sigma 4\pi R^2 + Q = 0$ it is that $Q = -\sigma 4\pi R^2$. If σ is positive, then Q must be negative.
Only for this Q, the electric field will be zero at any point outside of spherical surface.
On the other hand, the electric field can't be null inside the spherical surface. It will only occur if Q=0.

1.9 * Planes *y=-2* and *y=2* have surface densities of charge respectively, $1\ \mu C/m^2$ and $2\ \mu C/m^2$.
a) Calculate the difference of potential (d.d.p) between points *A(0,3,0)* and *B(0,5,0)*, as well as the work needed to move a $2\ \mu C$ charge from point *A* to point *B*. ¿Who is doing this work, the forces of the electric field or an external force against the electric field?
b) Calculate the difference of potential between points C(0,0,0) and D(0,1,-1) and the work needed to move a $-2\ \mu C$ point charge from point D to point C.

*Solution:*

a) $V_A - V_B = \int_A^B \vec{E}d\vec{r} = Ed = (\dfrac{1\cdot 10^{-6}}{2\varepsilon_0} + \dfrac{2\cdot 10^{-6}}{2\varepsilon_0})(5-3) = \dfrac{6\cdot 10^{-6}}{2\varepsilon_0} = \dfrac{3\cdot 10^{-6}}{8{,}85\cdot 10^{-12}} = 0{,}339\cdot 10^6\ V$

$W_{AB} = q(V_A - V_B) = 2\cdot 10^{-6}\cdot 0{,}339\cdot 10^6 = 0{,}678\ J$ As this work is positive, the work is done by the forces of electric field.

b) $V_C - V_D = \int_C^D \vec{E} d\vec{r} = \int_C^D (-\frac{1 \cdot 10^{-6}}{2\varepsilon_0}\vec{j})(dy\vec{j} - dz\vec{k}) = -\int_C^D \frac{1 \cdot 10^{-6}}{2\varepsilon_0} dy = -\frac{1 \cdot 10^{-6}}{2\varepsilon_0} = -56498\,V$

$W_{DC} = q(V_D - V_C) = -2 \cdot 10^{-6} \cdot 56498 = -0{,}113\,J$ As this work is positive, the work is done by the forces of electric field.

Unit 2: Conductors in electrostatic equilibrium. Dielectrics

2.1 Calculate the electric potential created by a conductor sphere (radius *R*) charged with charge *Q* at a point placed at a distance *r>R* from the centre of sphere. Calculate the electric potential of sphere.

*Solution:*

The electric field at any point outside of sphere comes from Gaus's law:

$$\phi = \int_{sphere} \vec{E} d\vec{S} = \int_{sphere} E dS = E \int_{sphere} dS = E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Therefore the electric potential at this point is

$$V = \int_r^{\infty} \vec{E} d\vec{r} = \int_r^{\infty} E dr = \int_r^{\infty} \frac{Q}{4\pi\varepsilon_0 r^2} dr = -\left.\frac{Q}{4\pi\varepsilon_0 r}\right|_r^{\infty} = \frac{Q}{4\pi\varepsilon_0 r}$$

The potential of sphere will be $V_{sphere} = V(r=R) = \frac{Q}{4\pi\varepsilon_0 R}$

2.2 Two conductor spheres with radii $R_1$ and $R_2$ ($R_1 > R_2$), the first one with charge and the second one without charge are joined with a conductor wire without capacitance (the electric influence between both spheres can be neglected). Calculate the charge and the electric potential of both spheres after the joining.

*Solution:*

Before both spheres are connected, the charge of first sphere is Q and its potential $\frac{Q}{4\pi\varepsilon_0 R_1}$

When both spheres are connected, then both spheres become the same conductor, and their potentials will be equal. To do it, the second sphere takes a part of charge Q coming from first sphere. If $Q_1$ and $Q_2$ are the charges of both spheres after the connection and V is their potential, must be verified that:

$Q_1 + Q_2 = Q$ and also that $V = \frac{Q_1}{4\pi\varepsilon_0 R_1} = \frac{Q_2}{4\pi\varepsilon_0 R_2}$

By solving this system of equations, it comes

$$Q_1 = \frac{QR_1}{R_1 + R_2} \qquad Q_2 = \frac{QR_2}{R_1 + R_2} \qquad V = \frac{Q}{4\pi\varepsilon_0 (R_1 + R_2)}$$

2.3 A point charge *q* is placed at a distance *d* from the centre of a conductor sphere with radius *R* and discharged (d>R). a) Calculate the potential of sphere. b) What if the sphere is linked to ground?

*Solution:*

a) If we calculate the potential at the centre of sphere: $V = \frac{q}{4\pi\varepsilon_0 d}$

b) If the sphere is linked to ground, its potential is null. V=0. To do it, the sphere takes some charge (Q) from ground. The potential of sphere due to both charges is:

$$V = \frac{q}{4\pi\varepsilon_0 d} + \frac{Q}{4\pi\varepsilon_0 R} = 0 \Rightarrow Q = -\frac{qR}{d}$$

2.4 A point charge *q* is placed at the centre of a hollow and conductor sphere (radii $R_1$ and $R_2$) discharged. Calculate the electric field at the different areas of the space: $r<R_1$, $R_1<r<R_2$, $r>R_2$. a) Calculate the electric potential of sphere. b) What if the sphere is linked to ground?

*Solution:*

a) The charge q induces a charge –q on inner surface of sphere, and then a charge q on outer surface of sphere.

$r<R_1$ $$\phi = E 4\pi r^2 = \frac{q}{\varepsilon_0} \Rightarrow E = \frac{q}{4\pi\varepsilon_0 r^2}$$

$R1<r<R_2$ E=0

$r>R_2$ $$\phi = E 4\pi r^2 = \frac{q-q+q}{\varepsilon_0} = \frac{q}{\varepsilon_0} \Rightarrow E = \frac{q}{4\pi\varepsilon_0 r^2}$$

The electric potential of sphere is $V = \int_{R_2}^{\infty} \vec{E} d\vec{r} = \int_{R_2}^{\infty} E dr = \int_{R_2}^{\infty} \frac{q}{4\pi\varepsilon_0 r^2} dr = \frac{q}{4\pi\varepsilon_0 R_2}$

b) If sphere is linked to ground, both the potential and the electric field in the sphere are null and also outside the sphere. To do it, all the charge on the outer surface of the sphere goes to ground through the ground linking. The outer surface density of charge is zero. Therefore:

$r>R_1$ ($R_1<r<R_2$ and $r\geq R_2$) E=0

$r<R_1$ $$\phi = E 4\pi r^2 = \frac{q}{\varepsilon_0} \Rightarrow E = \frac{q}{4\pi\varepsilon_0 r^2}$$

As the sphere is linked to ground, its potential is null.

2.5 * A hollow sphere (radii $R_1$ and $R_2$) is linked to ground. A point charge *q* is placed at the centre of sphere, and another point charge *Q* is placed outside of sphere at a distance *d* from its centre (d>R). Calculate the total charge of sphere. (In order to solve this exercise, apply the superposition principle taking in account the results of exercises 2.3 and 2.4).

*Solution:*

From exercise 2.3 (outer charge Q), the charge on outer surface of sphere is $-\frac{QR}{d}$ and the charge on inner surface is null.

From exercise 2.4 (charge q at the centre), the charge on outer surface is null, and charge on inner surface is –q.

By applying the superposition principle, the total charge of sphere is the charge due to both charges on both surfaces:

$$Q_{total} = -q - \frac{QR}{d}$$

2.6 A Van de Graaf generator is made up by a *10 cm* of radius conductor sphere. The generator is transferring electric charge to this sphere from a second smaller sphere having *5 cm* of radius. The distance between both spheres is *5 mm*. By assuming that electric field between the spheres is uniform (really it only changes around 13%) and that the dielectric breaking of the air is produced when the electric field reaches *1 KV/mm*, calculate the charge of each sphere when the spark is produced between the spheres.

*Solution:*

As the charge of big sphere comes from little sphere, both spheres have the same charge (but opposite sign): Q and –Q. Therefore the difference of potential between both spheres is (absolute value):

$$\left|V_{big} - V_{little}\right| = \frac{Q}{4\pi\varepsilon_0 10\cdot 10^{-2}} + \frac{Q}{4\pi\varepsilon_0 5\cdot 10^{-2}} = Q\cdot 27\cdot 10^{10}$$

On the other hand, this difference of potential must be $\left|V_{big} - V_{little}\right| = 1\cdot 5 = 5\ kV = 5000\ V$

By solving these equations $Q = \frac{5000}{27\cdot 10^{10}} = 0{,}02\ \mu C$

2.7 * A plane conductor (surface *S* and negligible thickness) is charged with a charge *Q*:

a) Let you say how the charge is distributed on conductor, and give its surface density of charge.

b) A second equal but discharged conductor plate is approached to the first one up to a little distance compared with the magnitude of both plates (we can then assume total influence between plates). Explain how the charges are distributed on both plates, and give their respective surface densities of charge.

*Solution:*

a) The charge Q will be distributed across both surfaces of conductor. Then $\sigma = \frac{Q}{2S}$

b) If there is total influence between both plates, the charge on that surface facing the first plate will be –Q/2 and then the charge on the other surface will be +Q/2. In this way, the net charge on second conductor will remain null and the electric field on both conductors is zero.

2.8 A capacitor (capacitance *C*) is connected to a power supply (d.d.p. V between its plates). Next is disconnected from power supply and it is connected to a second capacitor having capacitance *2C*, initially discharged.

a) Calculate the charge and potential of each capacitor after connecting them.

b) Plates of second capacitor are approached up to a distance a half of the initial distance. Compute the charge and the difference of potential between plates of each capacitor.

*Solution:*

a) The charge taken by the first capacitor is $Q = CV$ . After connecting a second capacitor, this charge is distributed between both capacitors, whose charges will be then $Q_1$ and $Q_2$. On the other hand, the difference of potential between the plates on both capacitors will be equal. These conditions can be written as:

$$Q_1 + Q_2 = Q \qquad \text{and} \qquad V' = \frac{Q_1}{C} = \frac{Q_2}{2C}$$

By solving this system: $Q_1 = \frac{Q}{3} = \frac{CV}{3}$ $\qquad Q_2 = \frac{2Q}{3} = \frac{2CV}{3}$ $\qquad V' = \frac{1}{3}V$

b) Now, the capacitance of second capacitor will be two times the initial capacitance, 4C. We could solve this case in the same way than on before, but we'll now use the equivalent capacitance: $C_{eq} = C + 4C = 5C$

The charge of equivalent capacitor will be the initial charge $Q = CV$ and then:

$$V'' = \frac{Q}{C_{eq}} = \frac{CV}{5C} = \frac{V}{5} \qquad Q'_1 = CV'' = \frac{1}{5}CV \qquad Q'_2 = 4CV'' = \frac{4}{5}CV$$

2.9 Repeat before exercise but keeping the power supply connected every time.

*Solution:*

In this case, the difference of potential between the plates of both capacitors will be V at any time. So

a) $Q_1 = CV$ $\qquad Q_2 = 2CV$ $\qquad V' = V$

b) $Q'_1 = CV$ $\qquad Q'_2 = 4CV$ $\qquad V'' = V$

2.10 Two capacitors (capacitance *2* and *3* $\mu F$) are connected in series to a *10 V* power supply. Compute the charge and the difference of potential between the plates of each capacitor:

a) Without using the idea of equivalent capacitance of set of capacitors.

b) By using the idea of equivalent capacitance of set of capacitors.

*Solution:*

a) As they are connected in series, the charges on both capacitors will be equal ($Q_1 = Q_2$). Moreover, the addition of potentials between the plates of both capacitors will be 10 V:

$$\frac{Q_1}{2\cdot 10^{-6}} + \frac{Q_2}{3\cdot 10^{-6}} = 10 \qquad Q_1 = Q_2$$

By solving this system: $\qquad Q_1 = Q_2 = 12\ \mu C$

And d.d.p.: $\qquad V_1 = \frac{Q_1}{C_1} = \frac{12}{2} = 6\,V \qquad V_2 = \frac{Q_2}{C_2} = \frac{12}{3} = 4\,V$

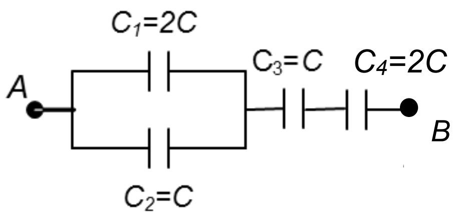
b) The equivalent capacitance of both capacitors is

$$\frac{1}{C_{eq}}=\frac{1}{C_1}+\frac{1}{C_2}=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}\Rightarrow C_{eq}=\frac{6}{5}\,\mu F$$

The charge $\quad Q_1=Q_2=C_{eq}\cdot 10=\frac{6}{5}\cdot 10\cdot 10^{-6}=12\mu C$

$$V_1=\frac{Q_1}{C_1}=\frac{12}{2}=6\,V \qquad\qquad V_2=\frac{Q_2}{C_2}=\frac{12}{3}=4\,V$$

2.11 Two capacitors ($C_1$=2C and $C_2$=C) are connected in parallel. This set is connected to two more capacitors ($C_3$=C and $C_4$=2C) in series as can be seen on picture. A voltage of 10 V is applied between points A and B. Calculate the charge, and the voltage (d.d.p.) on each capacitor, as well as the energy stored on each one and the total energy stored.




*Solution:*

This exercise can be solved in two different ways: without consider the equivalent capacitance, and by considering it. We will solve it in these two ways:

a) Without consider the equivalent capacitance:

As $C_1$ and $C_2$ are connected in parallel, $V_1=\frac{Q_1}{2C}=V_2=\frac{Q_2}{C}\Rightarrow Q_2=\frac{Q_1}{2}$ (1)

As $C_3$ and $C_4$ are connected in series, $Q_3=Q_4$ (2)

As the set of $C_1$ and $C_2$ is connected in series with $C_3$ and $C_4$, $Q_1+Q_2=Q_3=Q_4$ (3)

And the difference of potential between A and B: $V_1+V_3+V_4=\frac{Q_1}{2C}+\frac{Q_3}{C}+\frac{Q_4}{2C}=10$ (4)

We have now a system of four equations and four unknowns, $Q_1$ , $Q_2$ , $Q_3$ and $Q_4$ .

From eq. (1) and (3): $Q_1+\frac{Q_1}{2}=Q_3\Rightarrow Q_3=\frac{3Q_1}{2}$

And from eq. (4):

$$Q_1+2Q_3+Q_4=20C\Rightarrow Q_1+2\frac{3Q_1}{2}+\frac{3Q_1}{2}=20C\Rightarrow\frac{11}{2}Q_1=20C\Rightarrow Q_1=\frac{40}{11}C$$

Therefore: $\quad Q_2=\frac{20}{11}C \qquad Q_3=Q_4=\frac{60}{11}C$

and $\quad V_1=V_2=\frac{Q_1}{2C}=\frac{20}{11}V \qquad V_3=\frac{Q_3}{C}=\frac{60}{11}V \qquad V_4=\frac{Q_4}{2C}=\frac{30}{11}V$

Obviously is verified that: $V_1+V_3+V_4=\frac{20}{11}+\frac{60}{11}+\frac{30}{11}=\frac{110}{11}=10\,V$

Related to the stored energy:

$$W_1=\frac{Q_1^2}{2(2C)}=\frac{1}{4}\frac{1600}{121}C=\frac{400}{121}C \qquad W_2=\frac{Q_2^2}{2C}=\frac{200}{121}C \qquad W_3=\frac{Q_3^2}{2C}=\frac{1800}{121}C$$

$W_4=\frac{Q_4^2}{2(2C)}=\frac{900}{121}C$ Total stored energy: $W=W_1+W_2+W_3+W_4=\frac{3300}{121}=\frac{300}{11}C$

b) By taking in account the equivalent capacitance:
The equivalent capacitance of capacitors 1 and 2 (in parallel) is: $C_{12}=C_1+C_2=3C$

The equivalent capacitance of the whole set of capacitors is:

$$\frac{1}{C_{eq}}=\frac{1}{C_{12}}+\frac{1}{C_3}+\frac{1}{C_4}=\frac{1}{3C}+\frac{1}{C}+\frac{1}{2C}=\frac{11}{6C}\Rightarrow C_{eq}=\frac{6}{11}C$$

As $Q_3=Q_4$ and from the equivalent capacitance: $Q_4=Q_3=C_{eq}\cdot 10=\frac{6}{11}C\cdot 10=\frac{60}{11}C$

Voltages on capacitors 3 and 4 are: $V_3=\frac{Q_3}{C}=\frac{60}{11}V$ $\quad V_4=\frac{Q_4}{2C}=\frac{30}{11}V$

Voltage on capacitors 1 and 2 is: $V_1=V_2=10-(\frac{60}{11}+\frac{30}{11})=\frac{20}{11}V$

And charges on such capacitors: $Q_1=2C\cdot\frac{20}{11}=\frac{40}{11}C$ and $Q_2=C\cdot\frac{20}{11}=\frac{20}{11}C$

The total stored energy is: $W=\frac{Q_4^2}{2C_{eq}}=\frac{1}{2}\frac{3600\cdot C^2\cdot 11}{121\cdot 6C}=\frac{300}{11}C$

$$W_1=\frac{Q_1^2}{2(2C)}=\frac{400}{121}C \qquad W_2=\frac{Q_2^2}{2C}=\frac{200}{121}C \qquad W_3=\frac{Q_3^2}{2C}=\frac{1800}{121}C \qquad W_4=\frac{Q_4^2}{2(2C)}=\frac{900}{121}C$$

2.12 * The area of the plates of a parallel plate capacitor is S=50 $cm^2$, being d=2 mm the distance between them. The capacitor is charged with a charge Q=22 nC.

a) Find the capacitance of capacitor, its surface density of charge, the electric field between the plates, the difference of potential between them and the stored energy.

b) After the capacitor is charged, it is disconnected from power supply. What about the stored energy if the plates are approached or they are moved away? What if the capacitor is not disconnected from power supply?

*Solution:*

a) $C = \dfrac{\varepsilon_0 S}{d} = \dfrac{8{,}85 \cdot 10^{-12} \cdot 50 \cdot 10^{-4}}{2 \cdot 10^{-3}} = 22{,}1 \cdot 10^{-12}\ F = 22{,}1\ pF$

$\sigma = \dfrac{Q}{S} = \dfrac{22 \cdot 10^{-9}}{50 \cdot 10^{-4}} = 4{,}4 \cdot 10^{-6}\ C/m^2$     $E = \dfrac{\sigma}{\varepsilon_0} = \dfrac{4{,}4 \cdot 10^{-6}}{8{,}85 \cdot 10^{-12}} = 0{,}5 \cdot 10^6\ V/m$

$V = Ed = 0{,}5 \cdot 10^6 \cdot 2 \cdot 10^{-3} = 1000\, V$

$W = \dfrac{1}{2} C V^2 = \dfrac{1}{2} 22{,}1 \cdot 10^{-12} \cdot 10^6 = 11{,}05 \cdot 10^{-6}\ J$

b) If the plates are approached (d decreases), then the capacitance of capacitor ($\dfrac{\varepsilon_0 S}{d}$) increases. If the capacitor is disconnected from power supply, then the charge remains always constant, and the stored energy ($W = \dfrac{Q^2}{2C}$) decreases. Obviously, the force between the plates is attractive, because both plates have charges with different signs, and the work to approach the plates is done by the forces of the electric field. That's why the stored energy decreases.
If the plates are moved away, by repeating the before reasoning, the stored energy increases. Now, the work must be done by external forces, fighting against the attractive forces. The work done by these forces is stored by the capacitor.

If the capacitor remains connected to the power supply, then the voltage remains always constant. The stored energy ($W = \dfrac{1}{2} C V^2$) increases when the plates are approached, and decreases when the plates are moved away. It is important to note that now, as the power supply is connected to the capacitor, the power supply can take or give energy to the capacitor.

2.13 A capacitor with capacitance $C_0$ is connected to a power supply giving a difference of potential $V$. Next, a dielectric with relative dielectric permittivity $\varepsilon_r$ is inserted between the plates of capacitor. Compute the charge and the difference of potential of capacitor after inserting the dielectric. ¿Has the energy stored on capacitor increased or decreased when dielectric is inserted?

*Solution:*

After inserting the dielectric the new capacitance is $C' = \varepsilon_r C_0$ higher than $C_0$.

As the power supply isn't disconnected, the difference of potential on capacitor is constant. At any time $V' = V$ and $Q' = C'V' = \varepsilon_r C_0 V$

The energy stored before and after inserting the dielectric is:

$W = \dfrac{1}{2} C_0 V^2$     $W' = \dfrac{1}{2} C' V'^2 = \dfrac{1}{2} \varepsilon_r C_0 V^2 \quad > W$   Stored energy is increased

2.14 * A capacitor with capacitance $C_0$ is connected to a power supply giving a difference of potential $V$; capacitor is disconnected from power supply and a dielectric with relative dielectric permittivity $\varepsilon_r$ is inserted between the plates of capacitor. Compute the charge and the difference of potential of capacitor after inserting the dielectric. ¿Has the energy stored on capacitor increased or decreased when dielectric is inserted?

*Solution:*

Now, as the capacitor is disconnected after its charging, its charge will be constant:

$$Q' = Q = C_0 V \qquad V' = \frac{Q'}{C'} = \frac{C_0 V}{\varepsilon_r C_0} = \frac{V}{\varepsilon_r}$$

The energy stored before and after inserting the dielectric is:

$$W = \frac{1}{2} C_0 V^2 \qquad W' = \frac{1}{2} C' V'^2 = \frac{1}{2} \varepsilon_r C_0 \frac{V^2}{\varepsilon_r^2} = \frac{C_0 V^2}{2\varepsilon_r} < W$$

Stored energy decreases because the electric field between the plates of capacitor absorbs the dielectric due to its apparent density of charge. The work done by the electric field equals the decreasing on stored energy.

2.15 ** A capacitor with capacitance $C_0$ is filled with a dielectric having a relative dielectric permittivity *3*. Compute the new capacitance of capacitor

a) If it's filled only the left half of space between plates.

b) If it's filled only the lower half of space between plates.

*Solution:*

a) The initial capacitance is $C_0 = \frac{\varepsilon_0 S}{d}$

If only the left half of space is filled with a dielectric, the new capacitor can be considered as two capacitors in series. That on left with capacitance $C_1 = \frac{3\varepsilon_0 S}{\frac{d}{2}} = \frac{6\varepsilon_0 S}{d} = 6C_0$

And that on right with capacitance $C_2 = \frac{\varepsilon_0 S}{\frac{d}{2}} = \frac{2\varepsilon_0 S}{d} = 2C_0$

Then the equivalent capacitante is $\frac{1}{C'} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{6C_0} + \frac{1}{2C_0} = \frac{4}{6C_0} \Rightarrow C' = \frac{3}{2} C_0$

b) If only the lower half of space is filled with a dielectric, the new capacitor can be considered as two capacitors in parallel.

That below with capacitance $C_1 = \frac{3\varepsilon_0 \frac{S}{2}}{d} = \frac{3\varepsilon_0 S}{2d} = \frac{3}{2} C_0$

And that above with capacitance $C_2 = \frac{\varepsilon_0 \frac{S}{2}}{d} = \frac{\varepsilon_0 S}{2d} = \frac{1}{2} C_0$

Then the equivalent capacitante is $C'' = C_1 + C_2 = \frac{3}{2}C_0 + \frac{1}{2}C_0 = 2C_0$

Unit 3: Electric current

3.1 * A ring of radius *R* has a linear density of charge $\lambda$. If the ring turns with an angular speed $\omega$ around its axis, compute the intensity of current of ring.

*Solution:*

Let's consider a section of ring. Along an infinitesimal time *dt* the ring has turned an angle $d\varphi$. So, the angular speed is $\omega = \frac{d\varphi}{dt}$

Along *dt* the considered section has covered a distance $Rd\varphi$. The charge on this piece of ring is $dq=\lambda Rd\varphi$ As this charge has taken a time *dt* to pass through a cross section of ring, the intensity of current is

$$i = \frac{dq}{dt} = \frac{\lambda R d\varphi}{dt} = \lambda R\omega$$

3.2 Along a conductor of Cu with a radius of *1,3 mm* and length *1 m*, flows a *20 A* intensity of current. Compute the density of current J, the drift speed $V_d$, and the time taken by the electrons to cover *1 m*. Data: $n=1{,}806\cdot 10^{29}\ e^-/m^3$, $q = 1{,}6\cdot 10^{-19}\ C$.

*Solution:*

The area of cross section of conductor is $S = \pi R^2 = \pi(1{,}3\cdot 10^{-3})^2 = 5{,}3\cdot 10^{-6}\ m^2$

$$J = \frac{I}{S} = \frac{20}{5{,}3\cdot 10^{-6}} = 3{,}8\cdot 10^{6}\ A/m^2$$

$$v_d = \frac{J}{nq} = \frac{3{,}8\cdot 10^{6}}{1{,}806\cdot 10^{29} 1{,}6\cdot 10^{-19}} = 0{,}13\cdot 10^{-3}\ m/s$$

$$t = \frac{e}{v} = \frac{1}{0{,}13\cdot 10^{-3}} = 7{,}6\cdot 10^{3}\ s$$

3.3 Along a conductor of Cu with a radius of *1,3 mm* and length *1 m*, flows a *20 A* intensity of current. If the resistivity of Cu is $\rho_{Cu}=1{,}7*10^{-8}\ \Omega m$, compute the electric field E inside the conductor, its resistance R and the d.d.p. between its endings.

*Solution:*

$$E = \frac{J}{\sigma} = J\rho = 3{,}8\cdot 10^{6}\cdot 1{,}7\cdot 10^{-8} = 64{,}6\cdot 10^{-3}\ V/m$$

$$R = \frac{\rho L}{S} = \frac{1{,}7\cdot 10^{-8}\cdot 1}{5{,}3\cdot 10^{-6}} = 3{,}2\cdot 10^{-3}\ \Omega$$

$$V = IR = 20\cdot 3{,}2\cdot 10^{-3} = 64\cdot 10^{-3}\ V$$

3.4 Two *3* and *5* $\Omega$ resistors are connected in parallel; along the set of resistors is flowing a total intensity of current I = *10 A*. Find how much current flows along each resistor.

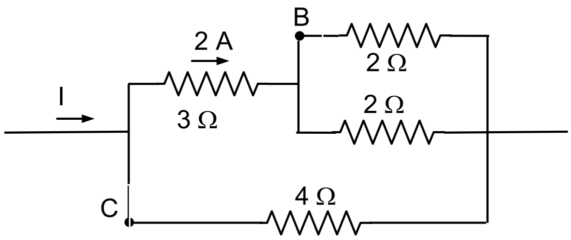
*Solution:*

If $I_3$ and $I_5$ are the intensities flowing along each resistor, we can write:

$$I_3 + I_5 = 10 \qquad \text{and} \qquad 3I_3 = 5I_5$$

By solving this system: $I_3 = \dfrac{25}{4}\,A$ $\qquad I_5 = \dfrac{15}{4}\,A$

3.5 Given the set of resistors on picture, find a) the total intensity I. b) If the set of resistors is isolated from any external circuit, compute its equivalent resistance between points B and C.




*Solution:*

a) The equivalent resistor of both 2 Ω resistors is 1 Ω. Therefore the potential difference between the endings of the whole set of resistors is $V = 2\cdot 3 + 2\cdot 1 = 8\,V$

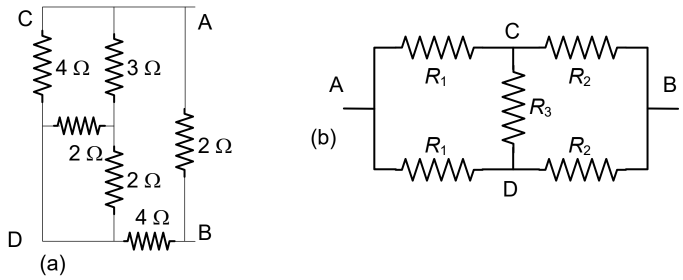
The intensity flowing along the 4 Ω resistor is $I_4 = \dfrac{8}{4} = 2\,A$

And the total intensity: $I = 2 + 2 = 4\,A$

b) If there is no external circuit, and we look at the circuit between points B and C, the set of two 2 Ω resistors in parallel is connected in series with the 4 Ω resistor. This association (5 Ω) is connected in parallel with the 3 Ω resistor. Therefore:

$$\frac{1}{R_{BC}} = \frac{1}{5} + \frac{1}{3} = \frac{3+5}{15} \Rightarrow R_{BC} = \frac{15}{8}\,\Omega$$

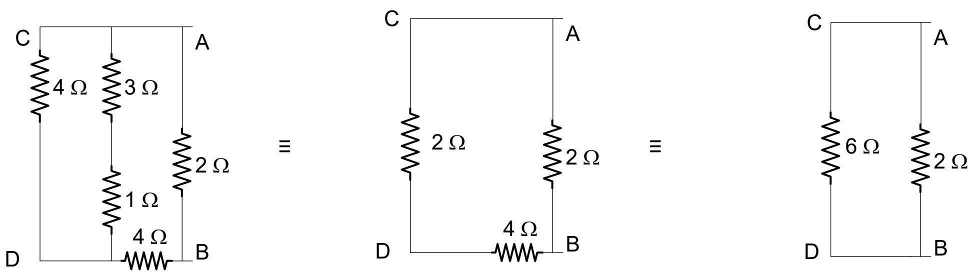
3.5 * For both circuits, compute the equivalent resistance between terminals *A* and *B* ($R_{AB}$) and between terminals *C* and *D* ($R_{CD}$).




*Solution:*

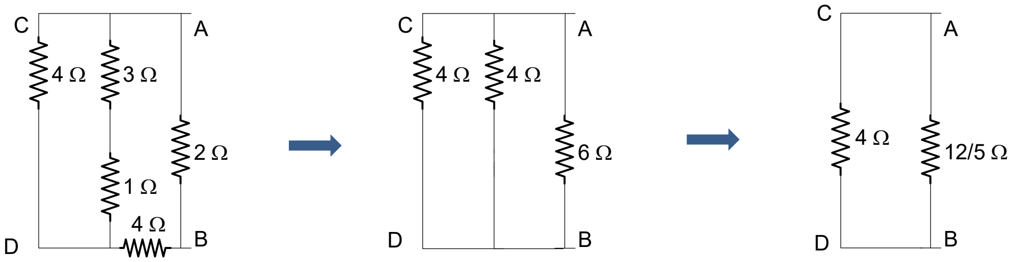
a) Between A and B:

Both 2 Ω resistors are connected in parallel, being 1 Ω their equivalent resistance. This equivalent resistance is connected in series with 3 Ω resistor, being the resulting circuit:



$$\text{Then } \frac{1}{R_{AB}} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \Rightarrow R_{AB} = \frac{3}{2}\Omega$$

Between C and D:



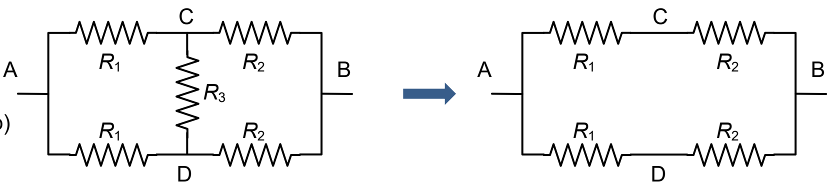

$$\text{Then } \frac{1}{R_{CD}} = \frac{1}{4} + \frac{5}{12} = \frac{8}{12} \Rightarrow R_{CD} = \frac{3}{2}\Omega$$

Even though $R_{AB}$ and $R_{CD}$ are equal, the equivalent resistance of a set of resistors between different points of a circuit usually leads to different values. The fact that $R_{AB}=R_{CD}$ on this example is only a coincidence.

b) Between A and B:

In this case, the resistors are associated neither in series nor in parallel. But if we pay attention on this circuit we can note that the upper and the lower part of circuit are equal, and then equal intensities will flow along upper and lower branches. For this reason, electric potential on points C and D are equal and then no intensity flows along $R_3$ resistor. As a consequence $R_3$ can be removed of circuit and $R_1$ and $R_2$ are connected in series both on upper branch as on lower branch.

(b)




$$\text{Therefore: } \frac{1}{R_{AB}} = \frac{1}{R_1+R_2} + \frac{1}{R_1+R_2} = \frac{2}{R_1+R_2} \Rightarrow R_{AB} = \frac{R_1+R_2}{2}$$

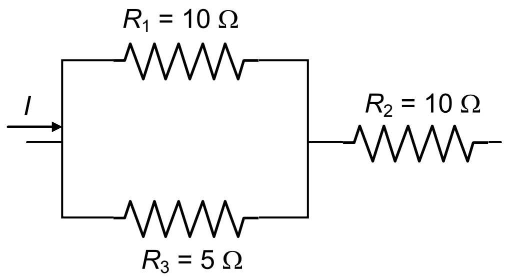
Between C and D:

In this case, both $R_1$ are connected in series and both $R_2$ are also connected in series. All of them connected in parallel with $R_3$. Therefore:

$$\frac{1}{R_{CD}} = \frac{1}{R_1+R_1} + \frac{1}{R_2+R_2} + \frac{1}{R_3} = \frac{1}{2R_1} + \frac{1}{2R_2} + \frac{1}{R_3} \Rightarrow R_{CD} = \frac{1}{\frac{1}{2R_1} + \frac{1}{2R_2} + \frac{1}{R_3}}$$

Unit 4: Energy and power

4.1 In the circuit of figure, justify:
a) Which resistor dissipates the highest power due to Joule heating?
b) Which resistor dissipates less power due to Joule heating?

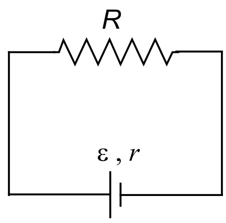



*Solution:*

a) The lost power on a resistor is $P = I^2 R$ . The total intensity is divided between $R_1$ and $R_2$ but the total intensity flows through $R_3$. Then $R_2$ and $R_3$ are the bigger resistors and $R_2$ is flowed by the higher intensity. So $R_2$ disipates more power than the other resistors.

b) The resistor dissipating less power should be $R_1$ or $R_3$. The lost power on a resistor is $P = \frac{V^2}{R}$ . As V is the same for both resistors, $R_1$ dissipates less power.

4.2 In the circuit of picture, ε = *6 V* and *r = 0,5 Ω.*. Lost power by Joule heating in *r* is *8 W*. Find:
a) The intensity of current on circuit.
b) The difference of potential between terminals of *R.*
c) *R*.
d) The generated power, the supplied power, and the efficiency of generator.
e) Verify that the supplied power equals the lost power by Joule heating on R.




*Solution:*

a) $8 = I^2 \cdot 0,5 \Rightarrow I = 4\ A$

b) $V_R = \varepsilon - Ir = 6 - 4 \cdot 0,5 = 4\ V$

c) $V_R = IR \Rightarrow 4 = 4 \cdot R \Rightarrow R = 1\Omega$

d) $P_g = \varepsilon I = 6 \cdot 4 = 24\ w$ $\qquad P_s = P_g - I^2 r = 24 - 4^2 \cdot 0,5 = 16\ w$

$$\eta_g = \frac{P_s}{P_g} = \frac{16}{24} = \frac{2}{3} = 0,66 \Rightarrow \eta_g = 66\ \%$$

e) $P_R = I^2 R = 16 \cdot 1 = 16\ w$

4.3 * A resistor *R* is connected to a generator with electromotive force ε and internal resistance *r*. Which should be the value of *R* in order the lost power on this resistor was the maximum?

*Solution:*

The current flowing along this circuit is $I = \frac{\varepsilon}{R + r}$

The lost power on R is $P_R = I^2 R = \frac{\varepsilon^2}{(R + r)^2} R$

The maximum of $P_R$ can be calculated by equalling its derivative to zero:

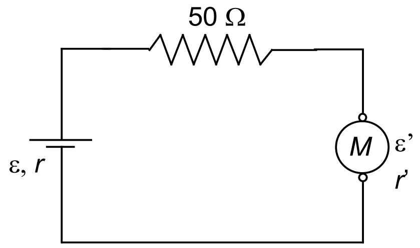
$$\frac{dP_R}{dR} = \frac{-\varepsilon^2 2(R + r)}{(R + r)^4} R + \frac{\varepsilon^2}{(R + r)^2} = \frac{-\varepsilon^2 2(R + r)R}{(R + r)^4} + \frac{\varepsilon^2 (R + r)^2}{(R + r)^4} = 0 \Rightarrow 2R = R + r \Rightarrow R = r$$

4.4 The engine of the drawn circuit consumes *50 W*, being a *20%* by Joule heating. If the generator supplies *100 W* to the circuit, compute:

a) Consumed power on 50 Ω resistor.

b) If the generator generates a power of 110 W, compute their characteristic parameters $\varepsilon$ and *r.*

c) The characteristic parameters of engine, $\varepsilon'$ and *r'*.




*Solution:*

a) $P_{50} = P_s - P_c = 100 - 50 = 50\ w$

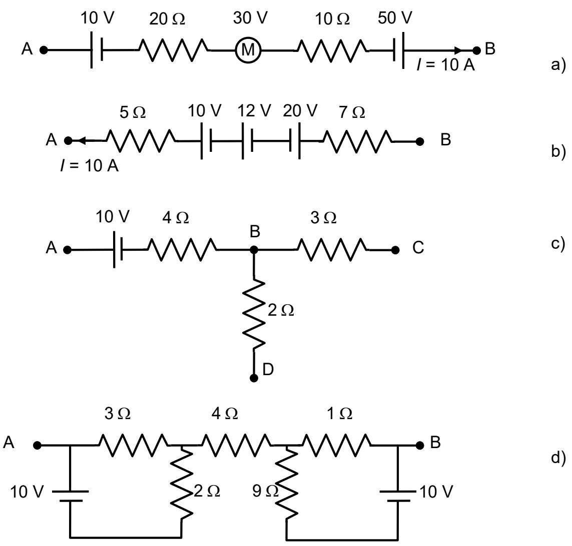
b) $P_{50} = 50 = I^2 50 \Rightarrow I = 1\ A$

$$110 = \varepsilon \cdot 1 \Rightarrow \varepsilon = 110\ V \quad P_r = 110 - 100 = 10 = I^2 r = r \Rightarrow r = 10\ \Omega$$

c) $P_{r'} = \dfrac{20 \cdot 50}{100} = 10 = I^2 r' = 1 \cdot r' \Rightarrow r' = 10\ \Omega$

$$P_t = \frac{80 \cdot 50}{100} = 40 = \varepsilon' I = \varepsilon' \cdot 1 \Rightarrow \varepsilon' = 40\ V$$

4.5 Compute the difference of potential between points *A* and *B* on next pictures:




*Solution:*

a) $V_A - V_B = 10(20 + 10) - (-10 - 30 + 50) = 300 - 10 = 290\ V$

b) $V_A - V_B = -10(5 + 7) - (-10 - 12 + 20) = -120 + 2 = -118\ V$

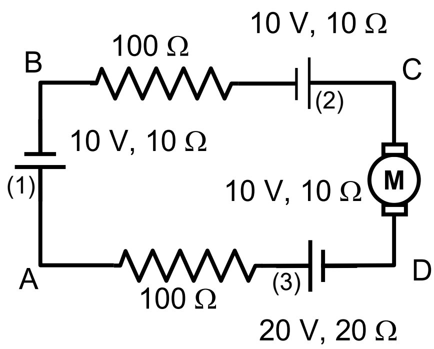
c) $V_A - V_B = 0 \cdot 4 - (-10) = 10\ V$

d) $V_A - V_B = 3 \cdot \dfrac{10}{5} - 1 \cdot \dfrac{10}{10} = 6 - 1 = 5\ V$

4.6 Given the circuit of figure:

a) Compute the magnitude and direction of intensity flowing along circuit.

b) Compute the difference of potential between points *A* and *C* ($V_A$-$V_C$), both along path *ABC* as along path *ADC*.



c) Which devices supply energy to the circuit? Compute the value of supplied power by each device.

d) Which devices consume energy from circuit? Compute the value of consumed power by each device.

e) Which are efficiencies of engine and power supply (2)?

f) If we modify the electromotive force of power supply (1), which must be its magnitude for the difference of potential between points *A* and *C* equals zero? Which is the intensity on circuit in this case?

*Solution:*

a) If we suppose the intensity flowing in clockwise direction:

$$I = \frac{20-10+10-10}{20+100+10+100+10+10} = \frac{10}{250} = 0{,}04\ A$$

b) $(V_A - V_c)_{ABC} = 0{,}04(10+100+10)-(-10+10) = 4{,}8\ V$

$(V_A - V_c)_{ADC} = -0{,}04(100+20+10)-(-20+10) = -5{,}2+10 = 4{,}8\ V$

c) Generators 2 and 3 are supplying energy to the circuit because the intensity flows from negative to positive terminal through them.

$$P_{S_2} = \varepsilon I - I^2 r = 10\cdot 0{,}04 - 0{,}04^2\cdot 10 = 0{,}384\ w$$

$$P_{S_3} = \varepsilon I - I^2 r = 20\cdot 0{,}04 - 0{,}04^2\cdot 20 = 0{,}768\ w$$

d) Generator 1, Engine and every resistor are consuming energy from circuit.

$$P_{c_1} = \varepsilon I + I^2 r = 10\cdot 0{,}04 + 0{,}04^2\cdot 10 = 0{,}416\ w$$

$$P_{c_{engine}} = \varepsilon' I + I^2 r' = 10\cdot 0{,}04 + 0{,}04^2\cdot 10 = 0{,}416\ w$$

$$P_{resistors} = I^2 \sum R_i = 0{,}04^2(100+100) = 0{,}32\ w$$

Of course, the supplied power must be equal to the consumed power:

$$P_S = 0{,}384 + 0{,}768 = 1{,}152\ w \qquad P_C = 0{,}416 + 0{,}416 + 0{,}32 = 1{,}152\ w$$

e) $\eta_{engine} = \dfrac{P_t}{P_c} = \dfrac{\varepsilon' I}{0{,}416} = \dfrac{0{,}4}{0{,}416} = 0{,}96 \approx 96\ \%$

$$\eta_2 = \frac{P_{s_2}}{P_{g_2}} = \frac{0{,}384}{\varepsilon I} = \frac{0{,}384}{10\cdot 0{,}04} = 0{,}96 \approx 96\ \%$$

f) If the difference of potential between A and C must be zero, then the intensity of current flowing along the circuit (supposed with the same direction) comes from:

$$(V_A - V_c)_{ADC} = -I(100+20+10)-(-20+10) = 0 \Rightarrow I = \frac{10}{130}\ A$$

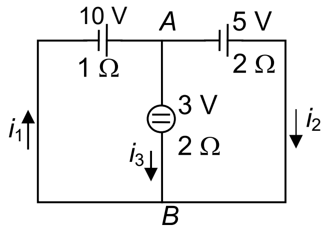
If we had supposed the opposite direction for I we would have obtained a negative intensity.

If we suppose now that the polarity of generator 1 is the same with the new electromotive force, its magnitude comes from:

$$(V_A - V_c)_{ABC} = \frac{10}{130}(10 + 100 + 10) - (-\varepsilon_1 + 10) = 0 \Rightarrow \varepsilon_1 = 0{,}77\,V$$

If we had supposed opposite polarity for generator 1, the new electromotive force would result negative.

Unit 5: Networks




5.1 In the circuit on picture, calculate the intensities in the three branches and the difference of potential between terminals of engine.

*Solution:*

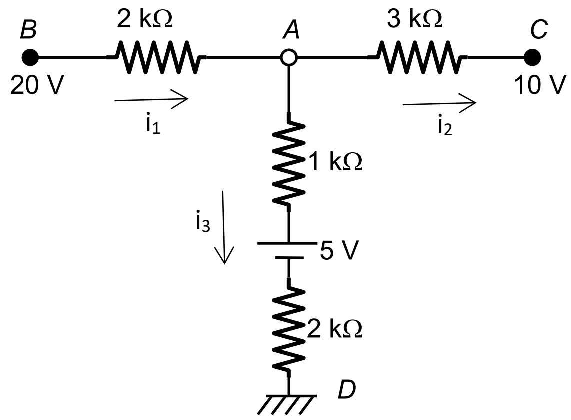
The direction assigned to $I_3$ means that the upper terminal of engine is the positive terminal, and the lower terminal is the negative. Then, writing Kirchoff's rule of junction A and that of loops for both loops:

$$\begin{aligned} i_1 &= i_2 + i_3 \\ i_1 1 + i_3 2 - (10-3) &= 0 \\ -i_3 2 + i_2 2 - (3+5) &= 0 \end{aligned}$$

By solving this system becomes: $i_1 = \frac{11}{2} A \qquad i_2 = \frac{19}{4} A \qquad i_3 = \frac{3}{4} A$

As $i_3$ is positive, it means that its supposed direction and then the polarity of engine are correct. Therefore

$$V_A - V_B = i_3 2 + 3 = \frac{3}{4} 2 + 3 = \frac{9}{2} V$$



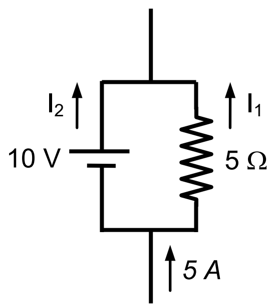

5.2 By using Kirchoff's rules, compute the potential on point *A* and the intensities of current along the branches of circuit on picture.

*Solution:*

If we name $i_1$ the intensity going from B to A, $i_2$ that going from A to C and $i_3$ the intensity from A to D, Kirchoff's rules become:

$$\begin{aligned} i_1 &= i_2 + i_3 \\ i_1 2 + i_3 3 - (-5) &= 20 \\ -i_2 3 + i_3 3 - (-5) &= 10 \end{aligned}$$

By solving this system becomes: $i_1 = \frac{75}{21} mA \qquad i_2 = \frac{20}{21} mA \qquad i_3 = \frac{55}{21} mA$




5.3 Along a wire flows an intensity of current *5 A*. This wire is divided into two branches: one of them with an ideal generator having a electromotive force *10 V*, and another branch with a *5 Ω* resistor. After these devices, both branches are joined.

a) Compute the intensity of current flowing along each branch.

b) Repeat the calculations after inverting the polarity of generator.

c) Repeat calculations of points a) and b) by changing the ideal generator by a real generator with internal resistance *10 Ω*.

*Solution:*

a) The difference of potential on resistor is defined by the battery, 10 V and then, being $i_1$ the intensity along the 5 Ω resistor (pointing to up): $5 \cdot i_1 = -10 \Rightarrow i_1 = -2 A$

It is, along the resistor flow 2 A to down. Therefore, along the battery flow 7 A to up.

b) If polarity of generator is inverted: $5 \cdot i_1 = 10 \Rightarrow i_1 = 2\ A$ to up. And along the battery flow 3 A to up.

c) If polarity of generator is that on picture must be verified:

$i_1 + i_2 = 5$ and $5 \cdot i_1 = 10 \cdot i_2 - 10$

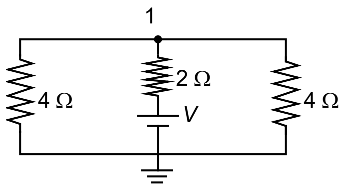
Solution of this system is: $i_1 = \dfrac{8}{3} A$ $\qquad i_2 = \dfrac{7}{3} A$

If polarity of generator is opposite to that on picture:

$i_1 + i_2 = 5$ and $5 \cdot i_1 = 10 \cdot i_2 + 10$

Solution of this system is: $i_1 = 4\ A$ $\qquad i_2 = 1\ A$

5.4 In the network on picture, find the voltage $V$ so that voltage on junction 1 was 50 V.

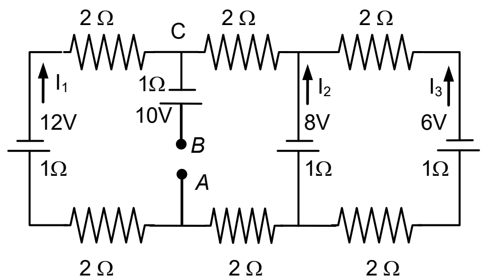



*Solution:*

If potential at point 1 is 50 V, then the intensities flowing along both 4 Ω resistors are 50/4=12,5 A each (going from 1 to ground). So, according junction's rule, along the middle branch must flow 25 A from ground to junction 1 and must be verified that:

$$50 = -25 \cdot 2 + V \Rightarrow V = 100\ V$$

5.5 Find the difference of potential between $A$ and $B$.




*Solution:*

With the intensities shown on picture:

$$i_1 + i_2 + i_3 = 0$$
$$i_1 9 - i_2 1 - (-12 + 8) = 0$$
$$i_2 1 - i_3 5 - (-8 + 6) = 0$$

By solving the system: $i_1 = -\dfrac{26}{59} A$ $\qquad i_2 = \dfrac{2}{59} A$ $\qquad i_3 = \dfrac{24}{59} A$

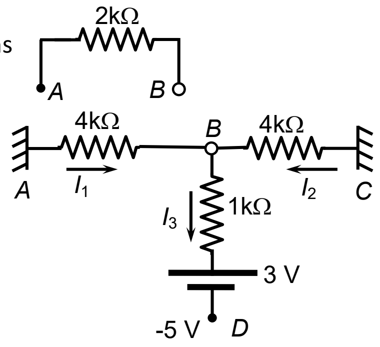
And $\quad V_A - V_B = i_1 5 - (-12 + 10) = -\dfrac{26}{59} \cdot 5 + 2 = \dfrac{-12}{59} \approx -0{,}20\ V$

5.6 Given the network on picture:

a) Compute the intensities of branches $I_1$, $I_2$, and $I_3$ by means of Kirchhoff's rules.

b) Find the Thevenin's equivalent generator between *A* and *B*, clearly showing its polarity.

c) In parallel to points *A* and *B* of network, a new 2 kΩ resistor is connected. Compute the intensity would flow along this resistor, clearly showing its direction.




*Solution:*

a) With the intensities shown on picture:

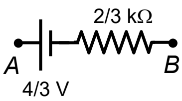
$$I_1 + I_2 = i_3$$

$$4I_1 + I_3 - (-3) = 0 - (-5) = 5$$

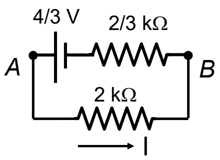
$$4I_1 - 4I_2 = 0$$

By solving the system: $I_1 = I_2 = \dfrac{1}{3}\,mA \qquad I_3 = \dfrac{2}{3}\,mA$

b) $\varepsilon_T = V_{AB} = 4I_1 = \dfrac{4}{3}\,V \qquad R_{eq} = \dfrac{2}{3}\,k\Omega$




c) $$I = \frac{\varepsilon_T}{R_{eq}+2} = \frac{\frac{4}{3}}{\frac{2}{3}+2} = \frac{1}{2}\,mA$$

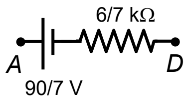



5.7 Compute the Thevenin's equivalent generator between points A and D on circuit of exercise 5.2. Which will be the intensity flowing along a new 5 kΩ resistor connected between A and D in the original circuit?

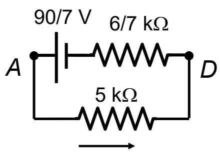
*Solution:*

With the intensities calculated on exercise 5.2 it comes:

$$\varepsilon_T = V_{AD} = 3I_3 + 5 = 3\frac{55}{21} + 5 = \frac{90}{7}\,V \qquad \frac{1}{R_{AD}} = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} = \frac{7}{6} \Rightarrow R_{AD} = \frac{6}{7}\,k\Omega$$



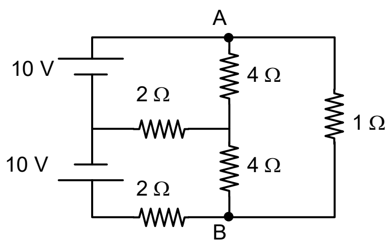

If a new 5 Ω resistor is connected between A and D, the resulting circuit, taking in account the Thevenin's equivalent generator computed is that on picture, and the current flowing along the resistor:




$$I = \frac{\varepsilon_T}{R_{eq}+5} = \frac{\frac{90}{7}}{\frac{6}{7}+5} = \frac{90}{41}\,mA$$

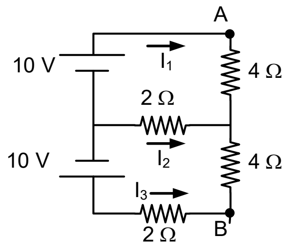
I

5.8.* In network of figure, find the intensity flowing along resistor *R* = 1 Ω, applying Thevenin's theorem between *A* and *B*.




*Solution:*

On first, we remove the 1 Ω resistor. The new circuit is:




$$I_1 + I_2 + i_3 = 0$$

Whose solution is $\quad 4I_1 - 2I_2 - 10 = 0$

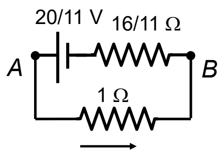
$$2I_2 - 6I_3 - (-10) = 0$$

$$I_1 = \frac{15}{11}A \qquad I_2 = -\frac{25}{11}A \qquad I_2 = \frac{10}{11}A \qquad \varepsilon_T = V_{AB} = 4I_1 - 4I_3 = \frac{20}{11}V$$

The equivalent resistance between A and B is:

$$\frac{1}{R_{AB}} = \frac{1}{2} + \frac{1}{\dfrac{1}{\frac{1}{2}+\frac{1}{4}} + 4} = \frac{1}{2} + \frac{1}{\frac{4}{3}+4} = \frac{1}{2} + \frac{3}{16} = \frac{11}{16} \Rightarrow R_{AB} = \frac{16}{11}\Omega$$

If we add the 1 Ω resistor between A and B, we have the original circuit:

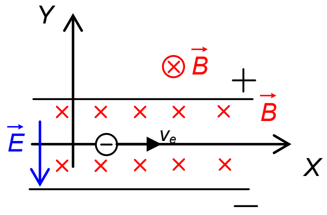



And the intensity flowing the 1 Ω resistor from A to B is:

$$I = \frac{20/11}{16/11+1} = \frac{20/11}{27/11} = \frac{20}{27}A$$

Unit 6: Magnetic forces

6.1 A bundle of electrons move between the plates of a capacitor with a difference of potential *V*. Between plates there is a uniform magnetic field perpendicular to the electric field. If plates of capacitor are separated a distance *d*, calculate the speed of the electrons not deflecting when they move between plates.

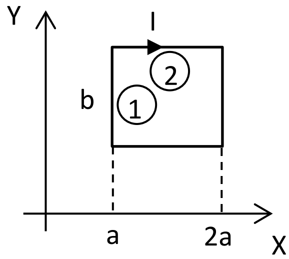



*Solution:*

Electrons not deflected will be those electrons with such speed ($v_e$) that electric force (pointing to up) cancels magnetic force (pointing to down): $qE = qv_eB$

As $E = \dfrac{V}{d}$ therefore: $q\dfrac{V}{d} = qv_eB \Rightarrow v_e = \dfrac{V}{Bd}$

6.2 Let's consider the rectangular loop on picture, with sides *a* and *b*, and flowed by an intensity *I* in the shown direction. The loop is inside a no uniform magnetic field $\vec{B} = B_0\dfrac{a}{x}\vec{k}$. Calculate the forces acting on sides 1 and 2.




*Solution:*

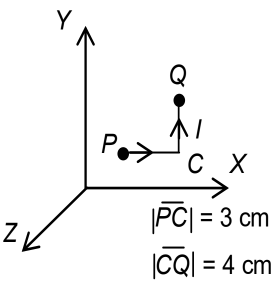
Along side 1, magnetic field is uniform $\vec{B}_1 = B_0\dfrac{a}{a}\vec{k} = B_0\vec{k}$

and the force: $\vec{F}_1 = Ib\vec{j} \times B_0\vec{k} = IbB_0\vec{i}$

Along side 2, magnetic field is not uniform, and then we'll consider an infinitesimal piece of conductor $d\vec{l} = dx\vec{i}$ The force acting over this piece of conductor is: $d\vec{F}_2 = Idx\vec{i} \times B_0\dfrac{a}{x}\vec{k} = -IB_0\dfrac{a}{x}dx\vec{j}$

The force over side 2 will be its integral: $\vec{F}_2 = -\int_a^{2a} IB_0\dfrac{a}{x}dx\vec{j} = -IB_0 a\ln 2\,\vec{j}$

6.3 By the segment of conductor in the figure flows a current *I* = 2 A from *P* to *Q*. It exists a magnetic field $\vec{B} = 1\vec{k}\,T$. Find the total force acting on conductor and prove that it is the same that if the entire conductor was a straight segment from *P* to *Q*.




*Solution:*

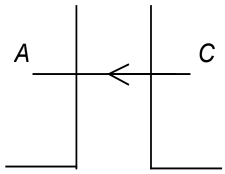
The total force acting on conductor will the force acting on segment PC plus that acting on segment CQ:

$$\vec{F} = \vec{F}_{PC} + \vec{F}_{CQ} = 2\cdot 3\cdot 10^{-2}\vec{i} \times \vec{k} + 2\cdot 4\cdot 10^{-2}\vec{j} \times \vec{k} = (-6\vec{j} + 8\vec{i})10^{-2}\ N$$

If the entire conductor was a straight segment from P to Q, the force would be the same:

$$\vec{F}_{PQ} = 2(3\cdot 10^{-2}\vec{i} + 4\cdot 10^{-2}\vec{j}) \times \vec{k} = (-6\vec{j} + 8\vec{i})10^{-2}\ N$$

6.4 Along conductor *AC* of figure flows a current of 10 A (it's a part of an electric circuit), being able to glide along two vertical rods.
Compute the necessary uniform magnetic field, perpendicular to the plane of the figure, in order that the magnetic force on conductor could equilibrate the gravitational force. Which should be the direction of magnetic field?
The length of conductor is 10 cm and its mass, 20 g.

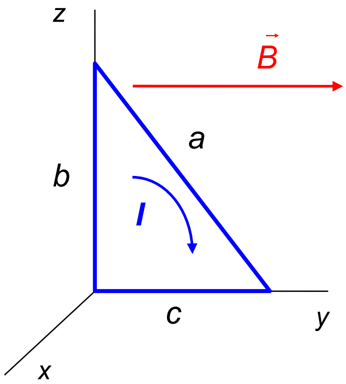



*Solution:*

In order the magnetic field can cancel the gravitational force, the magnetic field must exit from paper to the reader. Its magnitude has to verify that:

$$ILB = mg \Rightarrow 10 \cdot 10 \cdot 10^{-2} B = 20 \cdot 10^{-3} \cdot 9{,}8 \Rightarrow B = 0{,}196\ T$$

6.5 Along the loop of the figure of sides *a*, *b* and *c*, flows an intensity *I* in the shown direction. The loop is placed inside a magnetic field $\vec{B} = B\vec{j}$.
Find:
a) Magnetic forces on sides of the loop.
b) Magnetic moment of the loop.
c) Torque acting on loop.




*Solution:*

a) $\vec{F}_b = Ib\vec{k} \times B\vec{j} = -IbB\vec{i}$ $\qquad$ $\vec{F}_c = Ic\vec{j} \times B\vec{j} = 0$

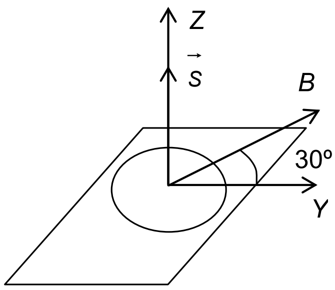
$\vec{F}_a = I(-b\vec{k} + c\vec{j}) \times B\vec{j} = IbB\vec{i}$

Obviously, as magnetic field is uniform, total force is null.

b) $\vec{m} = I\vec{S} = -\dfrac{Ibc}{2}\vec{i}$

c) $\vec{\tau} = \vec{m} \times \vec{B} = -\dfrac{Ibc}{2}\vec{i} \times B\vec{j} = -\dfrac{IBbc}{2}\vec{k}$

6.6 In order to measure a magnetic field, a coil with 200 turns of 14 $cm^2$ of area and making an angle of 30º with the field is used. When flowing an intensity of 0,7 A, a torque of $980 \cdot 10^{-6}$ Nm is measured. Compute *B*.




*Solution:*

The magnetic moment of the coil is $\vec{m} = NI\vec{S} = 200 \cdot 0{,}7 \cdot 14 \cdot 10^{-4}\vec{k} = 1{,}96 \cdot 10^{-1}\vec{k}\ A \cdot m^2$

And the modulus of the torque:

$$|\vec{\tau}| = |\vec{m} \times \vec{B}| = mBsen60º = 0{,}196 \cdot B \cdot \frac{\sqrt{3}}{2} = 980 \cdot 10^{-6} \Rightarrow B = 5{,}77 \cdot 10^{-3}\ T$$

In vector form: $\vec{B} = 5{,}77 \cdot 10^{-3}(\cos 30º\,\vec{j} + sen30º\,\vec{k}) = (5\vec{j} + 2{,}89\vec{k})10^{-3}\ T$

6.7 ** A conductor rod with length L moves with speed v in the plane of paper. Perpendicularly to this plane is acting a uniform magnetic field B into the page. In this situation, a difference of potential appears between the endings of the rod.

a) Could you explain why this difference of potential appears?

b) Find the magnetic force acting over an electron of the rod.

c) If the electric field appearing inside the rod is uniform, find its magnitude and the difference of potential between the endings of the rod.

*Solution:*

a) When the rod is moving inside the magnetic field, over every electron inside the rod acts a magnetic force given by $\vec{F}_m = q\vec{v} \times \vec{B}$ . The direction of this force push the electrons towards an ending of the rod, becoming this ending negatively charged. Therefore, the other ending of the conductor, with less negative charges becomes positively charged. The consequence is that an electric field appears inside the conductor, whose direction is that pf the rod, and a difference of potential appears between the endings of the rod.

b) The magnetic force acting over an electron, as it has been already said goes in the direction of the rod and its modules is $F_m = qvB$

c) The force acting over an electron due to the electric field inside the rod is If the electric field created by the distribution of charges inside the rod is $F_e = qE$ . This force cancels the magnetic force,in such way that

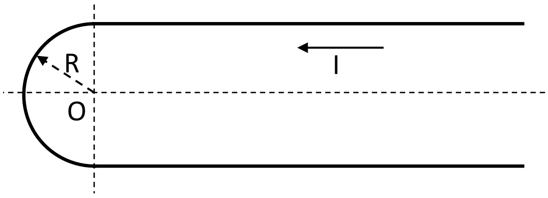
$$qvB = qE \Rightarrow E = vB$$

If E is uniform, then the difference of potential between the endings of the rod is

$$V = Ed = vBL$$

Unit 7: Sources of Magnetic field

7.1 Two parallel semi-infinite conductors are joined by a semi circumference, as can be seen on picture. Along the set of conductors flows an intensity of current I. Compute at point O (centre of semi circumference), always giving its direction:




a) The magnetic field produced by one of the straight conductors.
b) The magnetic field produced by the semi circumference.
c) The total magnetic field produced by the set of conductors.

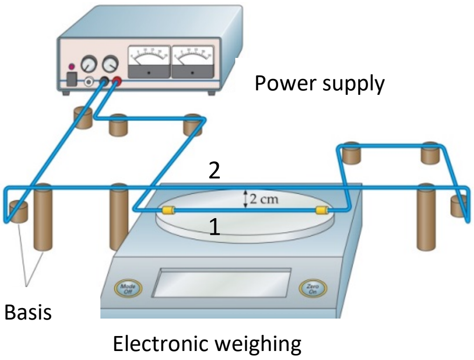
*Solution:*

a) $B = \frac{1}{2}\frac{\mu_0 I}{2\pi R}$ Perpendicular to paper and pointing to the reader

b) $B = \frac{1}{2}\frac{\mu_0 I}{2R}$ Perpendicular to paper and pointing to the reader

c) $B = 2\frac{1}{2}\frac{\mu_0 I}{2\pi R} + \frac{1}{2}\frac{\mu_0 I}{2R} = \frac{\mu_0 I}{4R}(\frac{2}{\pi} + 1)$

7.2 A current weighing scale is made up as is shown on picture:




An horizontal straight conductor 10 cm sized (1) is over the plate of an electronic weighing scale and connected to a second horizontal straight conductor (2), at a distance 2 cm from before conductor (the thickness of conductors can be neglected). Both conductors are connected to a D.C. power supply, making up a circuit. When the power supply is switched on, the reading of weighing scale increases 5.0 mg. (with respect to the reading with the power supply switched off).

a) Explain why the reading of weighing scale increases when power supply is switched on.
b) Compute the magnitude of intensity flowing along the circuit when power supply is switched on.
c) Red and black terminals of power supply correspond to its positive and negative terminals. ¿Which would be the reading of weighing scale if we invert the polarity of circuit? (it is, if the intensity would flow in opposite direction).
d) If the power supply can be taken as ideal, and it is supplying 12 V to the circuit, compute the lost power by Joule heating on all the resistors of circuit.

$\mu_0=4\pi\cdot10^{-7}$ (I.S. units)

*Solution:*

a) When power supply is switched on the intensity flows along conductors 1 and 2 in opposite direction. Therefore a rejecting force appears between both conductors and so a higher reading on weighing scale.
b) The rejecting force acting on each conductor is:

$$F = ILB = I10\cdot10^{-2}\frac{\mu_0 I}{2\pi 2\cdot10^{-2}} = I^2 10^{-6} = 5\cdot10^{-6}9{,}8 \Rightarrow I = \sqrt{49} = 7\ A$$

c) The reading of weighing scale would be the same, because if both intensities change their direction, the force between them is a rejecting force.

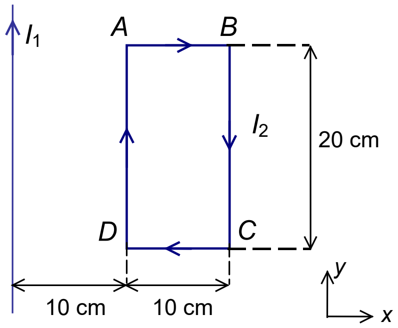
d) The lost power by Joule heating is $P_J = VI = 12 \cdot 7 = 84\,w$

7.3 An infinite and straight line conductor is flowed by an intensity $I_1 = 30\ A$. The rectangle *ABCD*, whose sides *BC* and *DA* are parallel to conductor is in the same plane than straight conductor, and it's flowed by an intensity $I_2 = 10\ A$. Compute:

a) The magnetic flux produced by $I_1$ through the rectangle.

b) The force acting on each side of rectangle because of the magnetic field created by $I_1$.

c) Should the resulting force acting on four sides of rectangle be null?




*Solution:*

a) The magnetic field produced by $I_1$ at a point over the surface of rectangle, at a distance x from conductor is:

$$\vec{B} = -\frac{\mu_0 I}{2\pi x}\vec{k} = -\frac{6 \cdot 10^{-6}}{x}\vec{k}$$

The magnetic flux entering on paper is:

$$\phi = \int_{10\cdot10^{-2}}^{20\cdot10^{-2}} \frac{6\cdot10^{-6}}{x} 20\cdot10^{-2}dx = 12\cdot10^{-7}\int_{10\cdot10^{-2}}^{20\cdot10^{-2}}\frac{dx}{x} = 12\cdot10^{-7}\ln 2\,Wb$$

b) $$\vec{F}_{AD} = I_2\vec{L}\times\vec{B} = 10\cdot 20\cdot10^{-2}\vec{j}\times(-\frac{\mu_0 30}{2\pi 10\cdot10^{-2}}\vec{k}) = -12\cdot10^{-5}\vec{i}\ N$$

$$\vec{F}_{BC} = I_2\vec{L}\times\vec{B} = 10\cdot(-20\cdot10^{-2}\vec{j})\times(-\frac{\mu_0 30}{2\pi 20\cdot10^{-2}}\vec{k}) = 6\cdot10^{-5}\vec{i}\ N$$

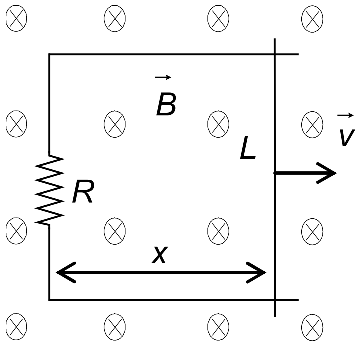
$$\vec{F}_{AB} = \int_{10\cdot10^{-2}}^{20\cdot10^{-2}} I_2 d\vec{x}\times\vec{B} = 10\int_{10\cdot10^{-2}}^{20\cdot10^{-2}}\frac{6\cdot10^{-6}}{x}dx\vec{j} = 6\cdot10^{-5}\ln 2\vec{j}\ N$$

$$\vec{F}_{CD} = \int_{10\cdot10^{-2}}^{20\cdot10^{-2}} I_2 d\vec{x}\times\vec{B} = -10\int_{10\cdot10^{-2}}^{20\cdot10^{-2}}\frac{6\cdot10^{-6}}{x}dx\vec{j} = -6\cdot10^{-5}\ln 2\vec{j}\ N$$

c) The resulting force shouldn't be null because even though the circuit is a loop, magnetic field over the loop isn't uniform. It can be proved because the summatory of forces along every side is not null:

$$\vec{F}_{TOTAL} = \vec{F}_{AD} + \vec{F}_{BC} + \vec{F}_{AB} + \vec{F}_{CD} = -6\cdot10^{-5}\vec{i}\ N \neq 0$$

Unit 9: Electromagnetic induction




9.1 A conductor rod with resistance negligible and length *L* glides without friction and constant speed *v* over a conductor *U* shaped. The U shaped conductor has a resistance *R* and it's placed inside a uniform magnetic field $\vec{B}$ perpendicular to conductor. Compute:

a) Magnetic flux crossing the loop as a function of *x*.

b) Induced current on the loop, showing its direction.

c) Force should act on the rod in order it be displaced at constant speed *v*.

d) Verify that the power produced by the force computed on c) is lost on resistance as Joule heating.

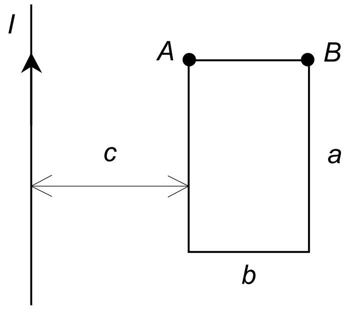
*Solution:*

a) As magnetic field is uniform $\phi = BLx$

b) $\varepsilon = \left|\frac{d\phi}{dt}\right| = BL\frac{dx}{dt} = BLv \Rightarrow i = \frac{\varepsilon}{R} = \frac{BLv}{R}$ Direction is counterclockwise

c) $F = iLB = \frac{BLv}{R}LB = \frac{B^2L^2v}{R}$

d) $P = Fv = \frac{B^2L^2v^2}{R}$ $P_{Joule} = i^2R = \frac{B^2L^2v^2}{R^2}R = \frac{B^2L^2v^2}{R}$




9.2 Along an infinite straight carrying current conductor flows an intensity *I* = *Kt* (*K* is a positive constant). A rectangular loop with sides *a* and *b* is placed in the same plane than the conductor, as can be seen on picture. Compute:

a) Induced e.m.f. on loop ε.

b) If the loop has a resistance *R*, compute the induced current *i*, showing its sense.

c) Magnetic force acting on side *AB* as a function of time, $F(t)$.

d) Mutual inductance coefficient *M* between conductor and loop.

*Solution:*

a) Magnetic field created by I at a point placed at a distance x from wire is:

$B = \frac{\mu_0 kt}{2\pi x}$ perpendicular to the paper and at points inside the loop, entering on paper.

Flux through the loop is: $\phi = \int_c^{c+b} \frac{\mu_0 kt}{2\pi x} a dx = \frac{\mu_0 kta}{2\pi} ln\frac{c+b}{c}$

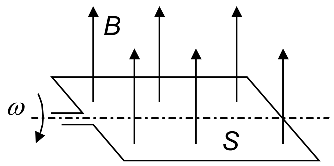
And the induced electromotive force: $\varepsilon = \left|\frac{d\phi}{dt}\right| = \frac{\mu_0 ka}{2\pi} ln\frac{c+b}{c}$

b) $i = \frac{\varepsilon}{R} = \frac{\mu_0 ka}{2\pi R} ln\frac{c+b}{c}$ Direction is counterclockwise

c) $F_{AB} = \int_c^{c+b} i\frac{\mu_0 kt}{2\pi x}dx = \frac{\mu_0 ka}{2\pi R} ln\frac{c+b}{c}\frac{\mu_0 kt}{2\pi}\int_c^{c+b}\frac{dx}{x} = (\frac{\mu_0 k}{2\pi} ln\frac{c+b}{c})^2 \frac{at}{R}$ perpendicular to AB and pointing to down

d) $M = \frac{\phi}{I} = \frac{\mu_0 a}{2\pi} ln \frac{c+b}{c}$

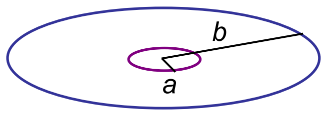
9.3 Compute the magnetic flux and the e.m.f. on the square loop of picture; this loop has an area *S* and it's turning at constant angular speed $\omega$ inside a uniform ans stationary magnetic field *B*.




*Solution:*

$$\phi = \vec{B} \cdot \vec{S} = BS\cos\omega t \qquad \varepsilon = \left| \frac{d\phi}{dt} \right| = BS\omega sen\omega t$$

9.4 * Two ring shaped loops, having radii *a* = 1 cm and *b* = 50 cm, are placed concentric in the same plane. If can be supposed that *a*<<*b* (magnetic field on loop *a* due to current on *b* can be supposed uniform) compute:




a) Mutual induction coefficient between both loops.
b) Magnetic flux across loop *b* when an intensity *I* = 5 A sized flows along *a*.

*Solution:*

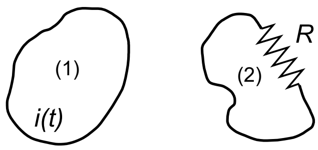
a) We'll suppose an intensity of current I flowing along big loop. Then, the magnetic field produced at points of little loop can be considered as uniform (remember that *a*<<*b*). Therefore:

$$\phi_a = BS = \frac{\mu_0 I}{2b}\pi a^2 \quad \text{and} \quad M = \frac{\phi_a}{I} = \frac{\mu_0 \pi a^2}{2b} = 4\pi^2 10^{-11}\ H$$

If we had considered an intensity flowing along little loop, then the supposition that the magnetic field is uniform at points of big loop is not accurate.

b) $\phi_b = MI = \frac{\mu_0 \pi a^2 I}{2b} = 2\pi^2 10^{-10}\ Wb$

9.5 The mutual inductance coefficient between circuits on picture is *M*. If a current $i(t) = I_0\cos(\omega t + \phi)$ flows along circuit 1, ¿which is the intensity flowing along circuit 2?




*Solution:*

The flux through circuit 2 is: $\phi_2 = Mi(t) = MI_0\cos(\omega t + \phi)$

And the intensity flowing along circuit 2: $i_2(t) = \frac{\varepsilon_2}{R} = \frac{1}{R}\left|\frac{d\phi_2}{dt}\right| = \frac{MI_0\omega}{R} sen(\omega t + \phi)$

9.6 Let's consider two coaxial coils having the same length $\ell$ but different cross section ($S_1$ and $S_2$) and different number of turns ($n_1$ and $n_2$). Coil 2 is placed inside coil 1 ($S_1 > S_2$).
a) By supposing an intensity of current $I_1$ flowing along coil 1, compute the magnetic flux through coil 2 and then the mutual inductance coefficient between both coils.
b) By supposing an intensity of current $I_2$ flowing along coil 2, compute the magnetic flux through coil 1 and then the mutual inductance coefficient between both coils.
c) Verify that mutual inductance coefficients computed on a) and b) matches.

*Solution:*

a) $\phi_2 = B_1 S_2 n_2 = \dfrac{\mu_0 I_1 n_1}{\ell} S_2 n_2$ $\qquad M = \dfrac{\phi_2}{I_1} = \dfrac{\mu_0 n_1}{\ell} S_2 n_2$

b) $\phi_1 = B_2 S_2 n_1 = \dfrac{\mu_0 I_2 n_2}{\ell} S_2 n_1$ $\qquad M = \dfrac{\phi_1}{I_2} = \dfrac{\mu_0 n_2}{\ell} S_2 n_1$

c) Both mutual inductance coefficients match.

9.7 Compute the self inductance coefficient of a coil having length $\ell$, cross section *S* and *n* turns.

*Solution:*

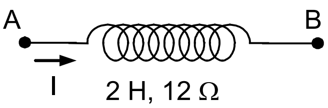
If we suppose an intensity of current I flowing along coil, the magnetic field inside the coil is:

$B = \mu_0 I \dfrac{n}{\ell}$ and the flux $\phi = BS = \mu_0 I \dfrac{n}{\ell} S$

Therefore the self-inductance coefficient is $L = \dfrac{\phi}{I} = \dfrac{\mu_0 n S}{\ell}$

9.8 Let's take a coil with a self-inductance coefficient *L = 2 H* and a resistance *R = 12 Ω.*

a) If an intensity of current *I=3t* (t is the time) is flowing from A to B, compute the difference of potential $V_A$-$V_B$ at time *t=1 s*.




b) If an intensity of current *I=10-3t* (t is the time) is flowing from A to B, compute the difference of potential $V_A$-$V_B$ at time *t=1 s*.

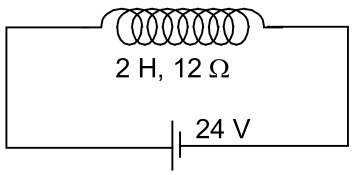
The coil is connected to an ideal generator with an electromotive force ε = 24 V (fig.(a)):

c) When the steady state is got, remaining constant the intensity of current, compute the intensity of current flowing along the circuit.

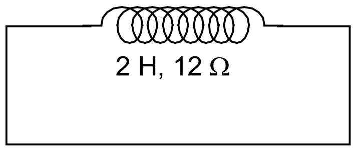
d) Compute the stored energy on coil.

e) If the coil is short-circuited and generator is removed (fig. (b)) ¿Which is the lost energy as heating on coil due to its resistance?

a)




b)




*Solution:*

a) $V_A - V_B = IR + L\dfrac{dI}{dt} = 3t \cdot 12 + 2 \cdot 3 = 36t + 6$

On time t=1 s $\quad (V_A - V_B)_1 = 42\,V$

b) $V_A - V_B = IR + L\dfrac{dI}{dt} = (10-3t)\cdot 12 + 2\cdot(-3) = -36t + 114$

On time t=1 s $\quad (V_A - V_B)_1 = 78\,V$

c) $I = \dfrac{24}{12} = 2\,A$

d) $W = \dfrac{1}{2} L I^2 = \dfrac{1}{2} 2 \cdot 2^2 = 4\,J$

e) $W = 4\,J$

Unit 10: Alternating current. Resonance and filters.

10.1 A resistor 5 Ω sized, an inductor 10 mH sized, and a capacitor 100 µF sized, are connected in series. The voltage on terminals of inductor is $u_L(t)=50\cos(500t+\frac{\pi}{4})\ \ V$. Compute the intensity of current flowing along three dipoles, the voltage on terminals of resistor and capacitor and the voltage on terminals of RLC dipole.

*Solution:*

The intensity of current (equal for three dipoles) can be calculated from voltage on terminals of inductor:

$$X_L=L\omega=10\cdot 10^{-3}\cdot 5\cdot 10^2=5\,\Omega \qquad I_m=\frac{U_{Lm}}{X_L}=\frac{50}{5}=10\,A$$

Phase lag on an inductor is π/2 (90º) : $\phi=90º=\phi_u-\phi_i=45-\phi_i\Rightarrow\phi_i=45-90=-45º$

Therefore: $I(t)=10\cos\left(500t-45º\right)A$

On capacitor: $X_C=\frac{1}{C\omega}=\frac{1}{100\cdot 10^{-6}\cdot 5\cdot 10^2}=20\,\Omega \qquad U_{Cm}=I_mX_C=10\cdot 20=200\,V$

$$\varphi=-90º=\varphi_u-\varphi_i=\varphi_u+45\Rightarrow\varphi_u=-90-45=-135º \qquad u_C(t)=200\cos\left(500t-135º\right)V$$

On resistor: $U_{Rm}=I_mR=10\cdot 5=50\,V \qquad u_R(t)=50\cos\left(500t-45º\right)V$

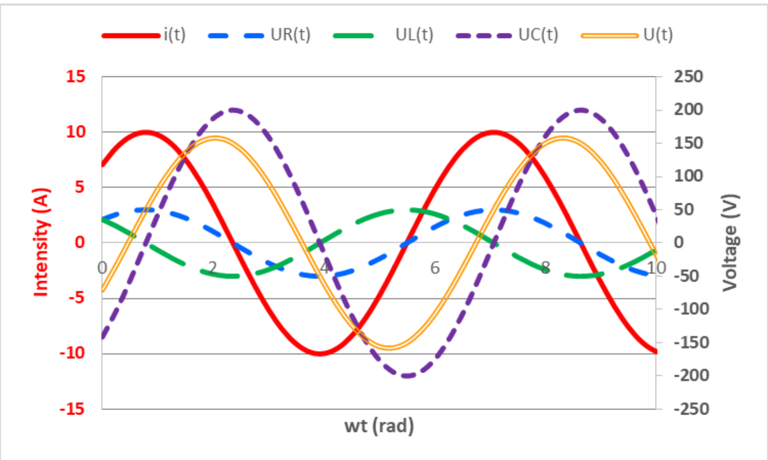
On the whole dipole (RLC): $Z=\sqrt{R^2+\left(X_L-X_C\right)^2}=\sqrt{5^2+(5-20)^2}=15{,}81\,\Omega$

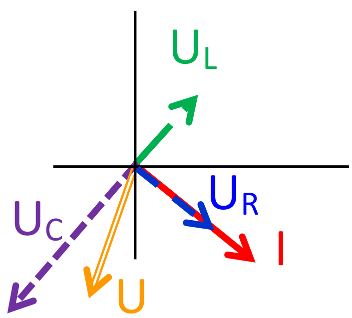
$$U_m=I_mZ=10\cdot 15{,}81=158{,}1V \quad tg\phi=\frac{X_L-X_C}{R}=\frac{5-20}{5}=-3\Rightarrow\phi=-71{,}6º$$

$$-71{,}6º=\phi_u-\phi_i=\phi_u+45\Rightarrow\phi_u=-71{,}6-45=-116{,}6º$$

Therefore: $u(t)=158{,}1\cos\left(500t-116{,}6º\right)V$

The drawing of intensities and voltages and the phasor diagram are:




It'll be very interesting the comparison between these results and those of exercise 10.3.

10.2 A circuit is made up by two basic dipoles in series. The terminals of this circuit are connected to an A.C. generator giving a voltage *u(t)*=150cos(500 *t*+10º) V, and flowing along the circuit an intensity of current *i(t)*=13,42cos(500*t*-53,4º) A. Determine the two basic dipoles and their magnitudes.

*Solution:*

The phase lag on dipole is: $\varphi = \varphi_u - \varphi_i = 10 - (-53,4) = 63,4º$

As phase lag is positive and not equal to 90º, it means that the dipoles on circuit are an inductor and a resistor.

$$tg63,4º = 2 = \frac{X_L}{R} \Rightarrow X_L = L\omega = 2R$$

$$Z = \frac{150}{13,42} = 11,18 = \sqrt{R^2 + X_L^{\,2}} = \sqrt{R^2 + 4R^2} = R\sqrt{5} \Rightarrow R = \frac{11,18}{\sqrt{5}} = 5\ \Omega$$

And $$L = \frac{2R}{\omega} = \frac{10}{500} = 20\ mH$$

10.3 Represent on a graph the drawing of intensity of current, voltage on resistor, voltage on inductor, voltage on capacitor and voltage on terminals of a *RLC* series circuit having a resistor 5 Ω sized, a 10 mH inductor and a 100 µF capacitor. The amplitude of intensity is 10 A, initial phase of intensity can be taken zero, and as angular frequency must be used that corresponding to the resonant frequency. Also represent its phasor diagram.

*Solution:*

The resonant angular frequency is:

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{10 * 10^{-3} * 100 * 10^{-6}}} = 1000\ rad/s \qquad f_0 = \frac{\omega_0}{2\pi} = 159,2\ Hz$$

With this angular frequency, the amplitude and phase lag of voltage on every device is:

Resistor: $U_{Rm} = I_m R = 10 \cdot 5 = 50\ V \qquad \varphi = 0 \Rightarrow \varphi_u = \varphi_i = 0$

Inductor: $X_L = L\omega_0 = 10 \cdot 10^{-3} \cdot 10^3 = 10\,\Omega \qquad U_{Lm} = I_m X_L = 10 \cdot 10 = 100\ V$

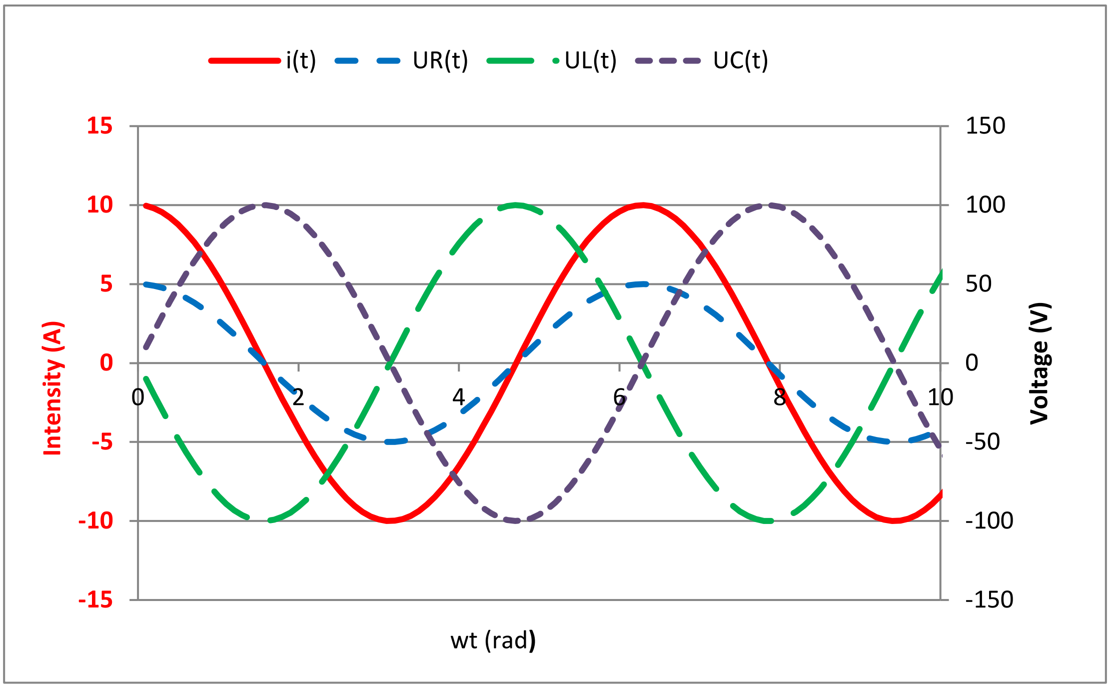
$$\varphi = 90º = \varphi_u - \varphi_i \Rightarrow \varphi_u = 90 + \varphi_i = 90º$$

Capacitor: $X_C = \dfrac{1}{C\omega_0} = \dfrac{1}{100 \cdot 10^{-6} \cdot 10^3} = 10\,\Omega \qquad U_{Cm} = I_m X_C = 10 \cdot 10 = 100\ V$

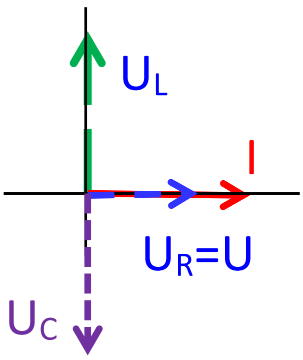
$$\varphi = -90º = \varphi_u - \varphi_i \Rightarrow \varphi_u = -90 + \varphi_i = -90º$$

RLC series circuit: $Z = \sqrt{R^2 + (X_L - X_C)^2} = R = 5\ \Omega \qquad \varphi = 0 = \varphi_u - \varphi_i \Rightarrow \varphi_u = 0$

Therefore, drawing of intensity and voltages is:




And the phasor diagram:




Note that at the resonant frequency, the voltages on terminals of inductor and capacitor are always equal but with different sign (phase lag 180º). Therefore the voltage on terminals of RLC dipole equals the voltage on resistor:

$$u_C(t) = -u_L(t) \qquad\qquad u(t) = u_R(t) = 50\cos(1000t)\,V$$

Note also that at the resonant frequency ($\omega_0$=1000 rad/s) Z=5 Ω, very low compared with that at 500 rad/s, Z=15,81 Ω. The consequence is that to reach the maximum current of 10 A, we must apply an AC with amplitude 158,1 V at 500 rad/s, but an amplitude of only 50 V at resonant frequency.

Moreover, you can see that on terminals of inductor or capacitor can be measured voltages even higher than those applied to the RLC dipole: by applying a voltage with amplitude 50 V can be measured amplitudes of 100 V on inductor and capacitor (at resonant frequency), or 200 V can be measured on capacitor against the applied 158,1 V at 500 rad/s. This behavior of RLC dipoles can be used to reach very high difference of potentials.

10.4 A *RLC* series circuit made up by L=2 H, C=2 µF and R=20 Ohm is connected to an adjustable frequency generator, giving an amplitude $U_m$=100 V.

a. Find the resonant frequency $f_0$.
b. Find the amplitude of intensity at resonant frequency.
c. Find the amplitudes of voltage on resistor, inductor and capacitor at resonant frequency.
d. Find the quality factor of circuit.
e. Find the bandwith of circuit.

*Solution:*

a) The resonant frequency is: $\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{2 \cdot 2 \cdot 10^{-6}}} = 500\, rad/s \Rightarrow f_0 = \frac{\omega_0}{2\pi} = 79{,}6\, Hz$

b) At the resonant frequency, the impedance of circuit equals the resistance. Therefore

$$I_m = \frac{U_m}{Z} = \frac{100}{20} = 5\, A$$

c) At the resonant frequency: $U_{mR} = I_m R = 5 \cdot 20 = 100\, V \qquad U_{mL} = I_m L\omega = 5 \cdot 2 \cdot 500 = 5000\, V = U_{mC}$

d) $Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{20}\sqrt{\frac{2}{2 \cdot 10^{-6}}} = 50$

e) The banwdith is the range of frequencies with a semiamplitude $f_2 - f_1 = \frac{R}{2\pi L} = \frac{20}{2\pi \cdot 2} = 1{,}6\, Hz$ around the resonant frequency (79,6 Hz). It is, the bandwith is the interval [78, 81,2] Hz.

Unit 11: Semiconductor materials

11.1 Find the density of electrons and holes on Ge on following circumstances:

a) Pure Ge at 300 K ($n_i$ (300 K) = 2,36·10$^{19}$ m$^{-3}$)

b) At 300 K doped with Sb (antimonium) with a concentration of $N_D$= 4·10$^{22}$ m$^{-3}$

c) At 300 K doped with In (indium) with a concentration of $N_A$= 3·10$^{22}$ m$^{-3}$

d) Pure Ge at 500 K ($n_i$ (500 K) = 2,1·10$^{22}$ m$^{-3}$)

e) At 500 K doped with Sb with a concentration of $N_D$=3·10$^{22}$ m$^{-3}$.

f) At 500 K doped with In with a concentration of $N_A$=4·10$^{22}$ m$^{-3}$

*Solution:*

a) As Ge is pure, $n = p = n_i = 2,36 \cdot 10^{19}\ e - h / m^3$

b) As concentration of donor impurities is much higher than intrinsic concentration (4·10$^{22}$ >>> 2,36·10$^{19}$) $n \approx N_D = 4 \cdot 10^{22}\ e/m^3$ $\quad p = \frac{n_i^2}{n} \approx 1,39 \cdot 10^{16}\ h/m^3$

c) As concentration of acceptor impurities is much higher than intrinsic concentration (3·10$^{22}$ >>> 2,36·10$^{19}$) $p \approx N_A = 3 \cdot 10^{22}\ h/m^3$ $\quad n = \frac{n_i^2}{p} \approx 1,86 \cdot 10^{16}\ e/m^3$

d) $n = p = n_i = 2,1 \cdot 10^{22}\ e - h / m^3$

e) At 500 K, the concentration of impurities and intrinsic concentration are of the same order and the density of electrons and holes must be calculated by solving the equation system:

$$\begin{cases} p \cdot n = n_i^2 \\ n = N_D + p \end{cases} \Rightarrow \begin{cases} p \cdot n = (2,1 \cdot 10^{22})^2 \\ n = 3 \cdot 10^{22} + p \end{cases}$$

whose solution is: $n = 4,08 \cdot 10^{22}\ e/m^3$ $\qquad p = 1,08 \cdot 10^{22}\ h/m^3$

f) $$\begin{cases} p \cdot n = n_i^2 \\ N_A + n = p \end{cases} \Rightarrow \begin{cases} p \cdot n = (2,1 \cdot 10^{22})^2 \\ 4 \cdot 10^{22} + n = p \end{cases}$$

whose solution is: $n = 2,1 \cdot 10^{22}\ e/m^3$ $\qquad p = 4,9 \cdot 10^{22}\ h/m^3$

11.2 An extrinsic n type semiconductor is made up by Si doped with 10$^{14}$ atoms of Sb/cm$^3$ (Sb is a donor impurity). The intrinsic density of Si at 300 K is $n_i$=1,5·10$^{10}$ cm$^{-3}$ and at 500 K $n_i$= 3,7 ·10$^{14}$ cm$^{-3}$(0,4 points each).

a) Find the density of electrons and holes on such semiconductor at 300 K.
b) Find the density of electrons and holes on such semiconductor at 500 K.
c) ¿Which would be the density of electrons and holes at 300K if the semiconductor wasn't doped?
d) If the mobilities of electrons and holes at 300 K are $\mu_n$ = 0,135 (m$^2$/Vs) and $\mu_p$ = 0,05 (m$^2$/Vs) respectively, and the charge of electron is $q_e = 1,6 \cdot 10^{-19}\ C$, compute the conductivity of semiconductor in case a).

e) Reason if the net electric charge of semiconductor, in the three cases, is positive, negative, or neutral.

*Solution:*

a) To find n and p, we have to solve the system of equations given by the mass action law and the electric netrality law:

$$\left.\begin{array}{r} n\cdot p = n_i^2 \\ N_A + n = N_D + p \end{array}\right\}$$

But in this case, as the doping is very high compared with the intrinsic density ($10^{14}$>>>$10^{10}$)

$$n \approx N_D = 10^{14}\ e^- / cm^3$$ and from the mass action law,

$$n\cdot p = 1{,}5^2\cdot 10^{20} \Rightarrow p = \frac{n_i^2}{n} = \frac{1{,}5^2\cdot 10^{20}}{10^{14}} = 2{,}25\cdot 10^6\ h/cm^3$$

b) Now, the density of donor impurities is of the same order than intrinsic density (≈$10^{14}$ in both cases), $N_A$ = 0 and $N_D$ = $10^{14}$ cm$^{-3}$. Therefore, by solving the system of equations given by the mass action law and the electric netrality law:

$$\left.\begin{array}{r} n\cdot p = 3{,}7^2\cdot 10^{28} \\ n = 10^{14} + p \end{array}\right\} \Rightarrow \left\{\begin{array}{l} n = 4{,}28\cdot 10^{14}\, e^- / cm^3 \\ p = 3{,}23\cdot 10^{14}\ h/cm^3 \end{array}\right.$$

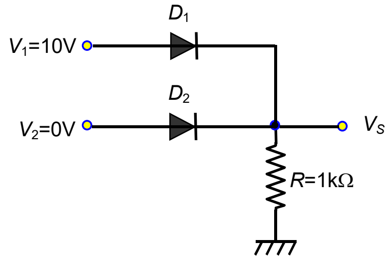
c) If the material is not dopped, the densities of electrons and holes are those of the intrinsic density: $n = p = n_i = 1{,}5\cdot 10^{10}$ cm$^{-3}$

d) The conductivity of a semiconductor comes from $\sigma = q_e\left(n\mu_n + p\mu_p\right)$:

$\sigma = q_e\left(n\mu_n + p\mu_p\right) \approx 1{,}6\cdot 10^{-19}\cdot 10^{14}\cdot 10^6\cdot 0{,}135 = 2{,}16\ \left(\Omega m\right)^{-1}$ (the term corresponding to the holes has been neglected because it is very low compared with the first one. Anyway, it is also correct its taking into account).

e) The electric charge of a semiconductor is neutral in any case, because of the electric neutrality law.

Unit 12: Semiconductor devices



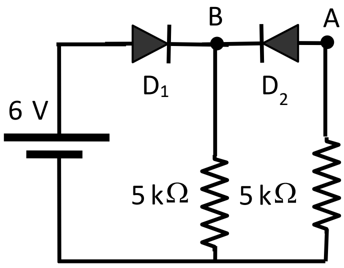

11.1 On circuit on picture, compute the voltage at output ($V_s$) as well as the intensities of current flowing along both diodes. Diodes $D_1$ and $D_2$ are diodes built with Si (drop forward voltage $V_u$=0,7 V).

*Solution:*

$D_1$ is forward biased and $D_2$ is reverse biased. Their internal resistances are neglected. Therefore

$$V_s = 10 - 0{,}7 = 9{,}3\,V \qquad I_1 = \frac{9{,}3}{1} = 9{,}3\,mA \qquad I_2 = 0$$

11.2 On circuit on picture both diodes have drop forward voltage $V_u$=0,7 V, and internal resistance can be neglected. Compute:




a) Intensities $I_1$ e $I_2$ flowing along diodes $D_1$ and $D_2$.
b) Potential difference $V_A$-$V_B$ between terminals of diode $D_2$.

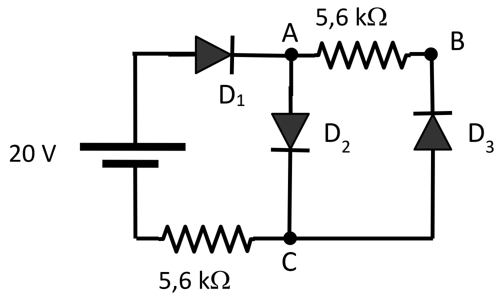
*Solution:*

a) $D_1$ is forward biased and $D_2$ is reverse biased. Internal resistances are neglected. Therefore

$$I_1 = \frac{6-0{,}7}{5} = \frac{5{,}3}{5} = 1{,}06\,mA \text{ clockwise direction } \quad I_2 = 0$$

b) $V_A - V_B = I_2 \cdot 5 - I_1 \cdot 5 = -5{,}3\,V$

11.3 On circuit on picture every diodes have drop forward voltage $V_u$=0,7 V, and internal resistance negligible (also for the battery).




a) On next table, mark with a cross the correct bias (forward or reverse) for every diode:

| Diode | Forward | Reverse |
|---|---|---|
| $D_1$ | | |
| $D_2$ | | |
| $D_3$ | | |

b) Compute intensities $I_1$, $I_2$ and $I_3$ flowing along diodes $D_1$, $D_2$ and $D_3$.
c) Compute the differences of potential $V_A$-$V_C$ and $V_C$-$V_B$.

*Solution:*

a)

| Diode | Forward | Reverse |
|---|---|---|
| $D_1$ | X | |
| $D_2$ | X | |
| $D_3$ | | X |

b) $I_1 = I_2 = \dfrac{20 - 0{,}7 - 0{,}7}{5{,}6} = 3{,}32\,mA \qquad I_3 = 0$

c) As $D_2$ is forward biased $V_A - V_C = 0{,}7\,V$

On the other hand, as $I_3$=0, $V_A - V_B = 0$ and therefore $V_C - V_B = V_C - V_A = -0{,}7\,V$